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Multi-soliton fusion phenomenon of Burgers equation and fission, fusion phenomenon of Sharma–Tasso–Olver equation

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Abstract

A direct rational exponential scheme is proposed to construct exact multi-soliton solutions and its fission, fusion phenomena after interaction of the solitons has been discussed. We have considered the Burgers and Sharma–Tasso–Olver equation as two concrete examples to show the fission and fusion of the solitary wave and the solitons, respectively. We improve different structured multi-soliton solutions with possible conditions for fission and fusion of the Burgers and the Sharma–Tasso–Olver equations arises in plasma physics and in ocean dynamics. The amplitude and velocity relations between solitons and/or solitary waves before and after interactions are given and a possible condition for fission and fusion is proposed. Furthermore, three-dimensional plots of the wave solutions are given to visualize the dynamics of the model.

Keywords: Direct rational exponential scheme; Burger equation; Sharma–Tasso–Olver equation; Traveling wave solutions; Multi-soliton.

1. Introduction

Nonlinear equations play an important task in applied mathematics, physics, biology and issues related to engineering due to their role in describing many real world phenomena. The world has witnessed a blooming number of serious facts like ocean acoustic and tsunami wave fields generated by earthquakes in 2011 in Japan along this coast known, mostly related to short-distance seismic activities like earthquakes from Mw 6.9 on 8.8 [1] and many types of acoustic wave in nuclear physics, in plasma physics, in ocean wave related phenomena. In these types of wave, the tallness and wavelength range are very important in terms of producing any disasters. If needed procedures are taken, these enormous powers can be twisted into a diverse energy sources which will be needed in different activities. Otherwise, to propulsion any system can be affected differently and may create disasters like "climate changes" and "global warming". The more wavelengths are up, the more they include dangerous for environment all around the world. To reduce harsh power of such giant disasters or to control them into useful energy sources, we should consider the mathematical structures of such natural problems. If we solve these problems by using different methods, we can find the best way of understanding such feasible disasters and then take needed safety.

The importance of obtaining exact solutions of nonlinear partial differential equations is still a big problem that compels scientists and engineers to seek different methods for exact solutions. In recent years, there has been an increase in interest in the study of the exact solutions of nonlinear equations which can be used to simulate many phenomena in different fields mentioned above. A variety of numerical and analytical methods have been developed to obtain accurate analytic solutions for problems, such as, inverse scattering transform [2], analytical methods [3], the exp-function method [4], the Hirota's bilinear method [5], the Jacobi elliptic function expansion method [6], the (G'/G)-expansion method [7–9], Backlund transformation [10],

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Darboux transformation [11], the multiple exp-function method [12], the symmetry algebra method [13], the Wronskian technique [14], the exp(-Φ(ξ))-expansion method [15,16] and few analytical methods [17–24]. Some advanced applications of new analytical methods for practical problems can be found [25–31]. Solitary and traveling wave solutions are investigated by some diverge researcher for different nonlinear models [32–37]. Studies of completely integrable equations and nonlinear phenomena are affluent in relation to solitary wave fields and engineering concepts. In soliton theory, non-elastic phenomena are rear case and there are rear model in the literature in which this phenomena exist. Actually, the interactions between two or more soliton solutions for integrable models are considered to be completely elastic and their amplitude, velocity, wave shape do not change after the non-linear interaction. Furthermore, some models exist in the literature are completely non-elastic, depending conditions between the wave vectors and velocities. Wazwaz [20–22] investigated multiple soliton solutions such type of non-elastic phenomena. Burgers equation and Sharma–Tasso–Olver equation are such types of model are studies in this article. Wang et al. [23] found non-elastic soliton fission and fusion: Burgers equation and Sharma–Tasso–Olver equations with only two dispersion relations. When studying ocean waves in general, large waves are of course more powerful and the wave power is determined by several parameters, such as wave height, wave speed, wavelength, and water density.

In this article, we investigate both elastic and non-elastic multi-soliton fission and fusion phenomena of the Burgers equation and Sharma–Tasso–Olver equations with few new dispersive relations. We investigate fusion phenomenon of the two solitary waves that two single solitary waves fusion to one (resonant) solitary wave after interaction of them and increases wave height with more potential energy can be used in turbine like mechanism. On the other hands, we investigate fission phenomenon of a solitary wave that one single solitary wave divided into two or more solitary waves after interaction and decreases wave height with smaller potential energy can be used in breaking seismic like waves.

2. Multi-soliton solution of Burgers equation and its fusion

In this section, we bring to bear a direct rational exponential approach to explain the completely non-elastic interaction clearly, the simplest non-linear $(1+1)$-dimensional Burger equation [17,23],

$$u_t + 2uu_x - u_{xx} = 0, \tag{1}$$

which has both the non-linear radiation and the diffusion effect.

For single soliton solution we first consider trial solution as

$$u(x,t) = r\frac{k_1 c_1 \exp(k_1 x + w_1 t)}{a_0 + c_1 \exp(k_1 x + w_1 t)}. \tag{2}$$

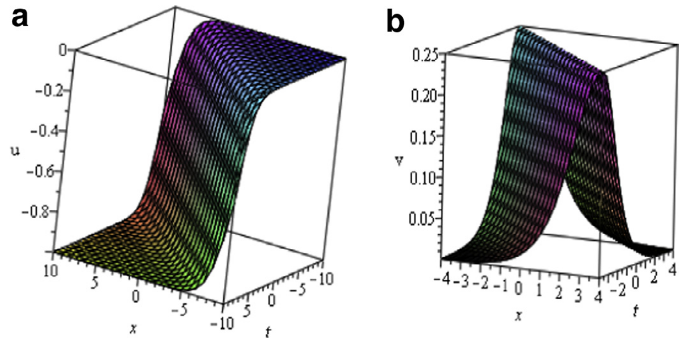

Fig. 1. (a) Profile of the single solitary wave solution Eq. (3) of Burger equation, (b) potential field Eq. (4) with $k_1 = c_1 = a_0 = 1$.

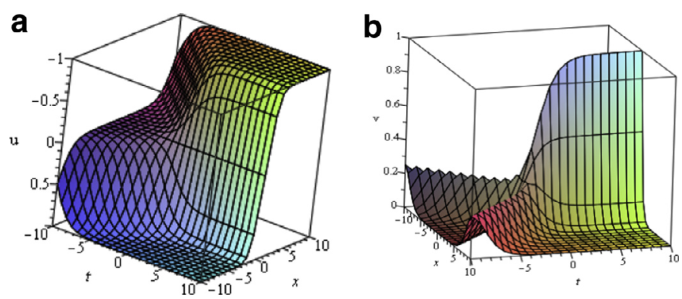

Fig. 2. (a) Profile of two solitary wave fusion solution Eq. (6) of Burger equation, (b) potential field Eq. (7) with $k_1 = 1, k_2 = -1, c_1 = c_2 = a_0 = 1$.

Inserting (2) and (1), and then maintenance the coefficients of $(\exp(k_1 x + w_1 t))^i$, $(i = \ldots -2, 1, 0, 1, 2 \ldots)$ is zero, yields a system of algebraic equations about a_0, c_1, w_1 and k_1 as follows:

$$c_1 w_1 + c_1 k_1^2 + 2rc_1 k_1^2 = 0, \quad w_1 a_0 - k_1^2 a_0 = 0.$$

Solving this over-determined system of algebraic equations for a_0, w_1, r with the aid Maple 13, we arrive at the following solutions: $a_0 = const$, $w_1 = k_1^2$, $r = -1$ and thus the solution is

$$u(x,t) = -\frac{k_1 c_1 \exp(k_1(x + k_1 t))}{a_0 + c_1 \exp(k_1(x + k_1 t))}, \tag{3}$$

and corresponding potential function is read as

$$v(x,t) = -\frac{k_1^2 c_1^2 \exp(2k_1(x + k_1 t))}{(a_0 + c_1 \exp(k_1(x + k_1 t)))^2} + \frac{k_1^2 c_1 \exp(k_1(x + k_1 t))}{a_0 + c_1 \exp(k_1(x + k_1 t))}. \tag{4}$$

To obtain two soliton solutions we just suppose

$$u(x,t) = r\left(\frac{k_1 c_1 \exp(\xi_1) + k_2 c_2 \exp(\xi_2) + a_{12}(k_1 + k_2) c_1 c_2 \exp\{\xi_1 + \xi_2\}}{a_0 + c_1 \exp(\xi_1) + c_2 \exp(\xi_2) + a_{12} c_1 c_2 \exp\{\xi_1 + \xi_2\}}\right), \tag{5}$$

where $\xi_1 = k_1 x + w_1 t$, $\xi_2 = k_2 x + w_2 t$ and the corresponding potential field reads $v = -u_x$. Directly inserting (5) in

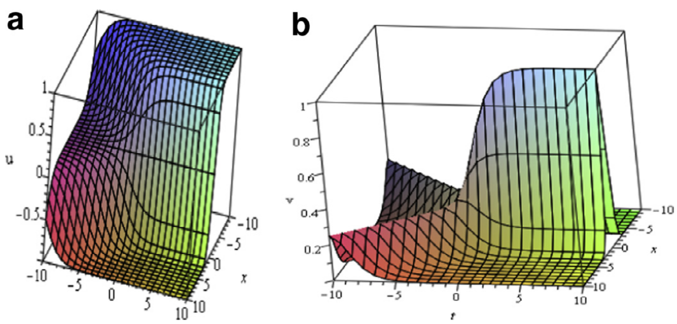

Fig. 3. (a) Profile of two solitary wave fusion solution Eq. (8) of Burger equation, (b) corresponding potential field v with $k_1 = 1, k_2 = -1, c_1 = c_2 = a_1 = 1$.

Eq. (1) via commercial software Maple-13, and collecting the coefficients of different power of exponential to zero, we attained a system of algebraic equation in terms of $r, k_1, k_2, w_1, w_1, c_1, c_2$ and a_{12}. Solving this system of algebraic equations for r, k_1, k_2, w_1, w_1 and a_{12} with the software, we gain the following solution of the unknown parameters.

Now according to the cases in the method we have

Set-1: $r = -1,\ a_0 = const,\ a_{12} = 0,\ w_1 = k_1^2,\ w_2 = k_2^2$ then

$$u(x,t) = -\frac{k_1c_1\exp(k_1(x+k_1t)) + k_2c_2\exp(k_2(x+k_2t))}{a_0 + c_1\exp(k_1(x+k_1t)) + c_2\exp(k_2(x+k_2t))}, \tag{6}$$

where a_0, c_1, c_2, k_1, k_2 are arbitrary constants.

The corresponding potential field reads

$$v = \frac{k_1^2c_1\exp(k_1(x+k_1t)) + k_2^2c_2\exp(k_2(x+k_2t))}{a_0 + c_1\exp(k_1(x+k_1t)) + c_2\exp(k_2(x+k_2t))} - \frac{(k_1c_1\exp(k_1(x+k_1t)) + k_2c_2\exp(k_2(x+k_2t)))^2}{(a_0 + c_1\exp(k_1(x+k_1t)) + c_2\exp(k_2(x+k_2t)))^2}. \tag{7}$$

From Fig. 2, which plots the fusion phenomenon of the two solitary waves with the parameters selected as $k_1 = 1,\ k_2 = -1$, we can clearly see that two single solitary waves fusion to one (resonant) solitary wave after interaction of them i.e., at a specific time $t = 0$. From careful analyses of Eqs. (6) and (7), it is conclude that for all the ranges of two arbitrary parameters k_1, k_2, only fusion occurs. Neither elastic scattering nor fission does exist.

Set-2: $r = -1,\ a_0 = 0,\ a_{12} = const,\ w_1 = k_1^2 + 2k_1k_2,\ w_2 = 2k_1k_2 + k_2^2$ then

$$u(x,t) = -\frac{k_1c_1\exp(\xi_1)+k_2c_2\exp(\xi_2)+a_{12}(k_1+k_2)c_1c_2\exp\{\xi_1+\xi_2\}}{c_1\exp(\xi_1)+c_2\exp(\xi_2)+a_{12}c_1c_2\exp\{\xi_1+\xi_2\}}, \tag{8}$$

where $\xi_1 = k_1x + (k_1^2 + 2k_1k_2)t,\quad \xi_2 = k_2x + (k_2^2 + 2k_1k_2)t$ and $c_1, c_2, k_1, k_2$are arbitrary constants.

The corresponding potential field reads $v = -u_x$.

The Fig. 3, which is also elastic scattering and no fission exist, but the fusion phenomenon of the two solitary waves exist for all the ranges of two arbitrary parameters k_1, k_2 like solution Eqs. (6) and (7).

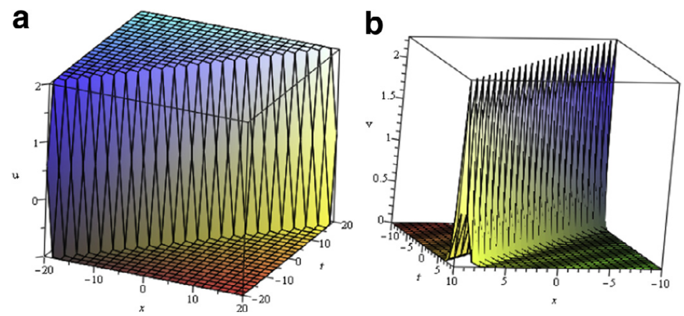

Fig. 4. (a) Profile of two solitary wave solution Eq. (9) of Burger equation, (b) corresponding potential field v with $k_1 = 1, k_2 = -2, c_1 = c_2 = w_1 = 1$.

Set-3: $r = -1,\ a_0 = 0,\ a_{12} = 0,\ w_1 = k_1^2 - k_2^2 + w_2,\ w_2 = const$ then

$$u(x,t) = -\frac{k_1c_1\exp(k_1x+(k_1^2-k_2^2+w_2)t)+k_2c_2\exp(k_2x+w_2t)}{c_1\exp(k_1x+(k_1^2-k_2^2+w_2)t)+c_2\exp(k_2x+w_2t)}, \tag{9}$$

where c_1, c_2, k_1, k_2, w_2 are arbitrary constants.

The corresponding potential field reads $v = -u_x$.

From Fig. 4 solution Eq. (9) is completely elastic, before and after collision of the two solitary waves their shape and size remain same. So, elastic scattering, no fusion and no fission exist for any values of the parameters.

To obtain three soliton solutions we just suppose

$$u(x,t) = r\left(\frac{\begin{array}{l}k_1c_1\exp(\xi_1)+k_2c_2\exp(\xi_2)+k_3c_3\exp(\xi_3)+a_{12}(k_1+k_2)c_1c_2.\\ \exp\{\xi_1+\xi_2\}+a_{23}(k_2+k_3)c_2c_3\exp\{\xi_2+\xi_3\}+a_{13}(k_1+k_3)c_1c_3\\ \exp\{\xi_1+\xi_3\}+a_{123}(k_1+k_2+k_3)c_1c_2c_3\exp\{\xi_1+\xi_2+\xi_3\}\end{array}}{\begin{array}{l}a_0+c_1\exp(\xi_1)+c_2\exp(\xi_2)+c_3\exp(\xi_3)+a_{12}c_1c_2\exp\{\xi_1+\xi_2\}\\ +a_{23}c_2c_3\exp\{\xi_2+\xi_3\}+a_{13}c_1c_3\exp\{\xi_1+\xi_3\}\\ +a_{123}c_1c_2c_3\exp\{\xi_1+\xi_2+\xi_3\}\end{array}}\right), \tag{10}$$

where $\xi_1 = k_1x + w_1t,\ \ \xi_2 = k_2x + w_2t,\ \ \xi_3 = k_3x + w_3t$ and the corresponding potential field reads $v = -u_x$.

Directly inserting (10) in Eq. (1) via commercial software Maple-13, and collecting the coefficients of different power of exponential to zero, we attained a system of algebraic and solving the system of algebraic equations via software, we gain the following solution of the unknown parameters.

Set-1: $r = -1,\ a_0 = const,\ a_{12} = 0, a_{23} = 0, a_{13} = 0, a_{123} = 0,\ w_1 = k_1^2,\ w_2 = k_2^2, w_3 = k_3^2$ then

$$u(x,t) = -\frac{k_1c_1\exp(k_1(x+k_1t))+k_2c_2\exp(k_2(x+k_2t))+k_3c_3\exp(k_3(x+k_3t))}{a_0+c_1\exp(k_1(x+k_1t))+c_2\exp(k_2(x+k_2t))++c_3\exp(k_{30}(x+k_3t))}, \tag{11}$$

where $a_0, c_1, c_2, c_3, k_1, k_2, k_3$ are arbitrary constants.

The corresponding potential field reads

$$v = \frac{k_1^2c_1\exp(k_1(x+k_1t))+k_2^2c_2\exp(k_2(x+k_2t))+k_3^2c_3\exp(k_3(x+k_3t))}{a_0+c_1\exp(k_1(x+k_1t))+c_2\exp(k_2(x+k_2t))+c_3\exp(k_3(x+k_3t))} - \frac{(k_1c_1\exp(k_1(x+k_1t))+k_2c_2\exp(k_2(x+k_2t))+k_3^2c_3\exp(k_3(x+k_3t)))^2}{(a_0+c_1\exp(k_1(x+k_1t))+c_2\exp(k_2(x+k_2t))+c_3\exp(k_3(x+k_3t)))^2}. \tag{12}$$

Set-2: $r=-1$, $a_0=a_{12}=a_{23}=a_{13}=0$, $a_{123}=const$, $w_1=k_1^2+k_1k_2+k_2k_3+k_1k_3$, $w_2=k_2^2+k_1k_2+k_2k_3+k_1k_3$, $w_3=k_3^2+k_1k_2+k_2k_3+k_1k_3$ then,

$$u(x,t)=-\left(\frac{k_1c_1\exp(\xi_1)+k_2c_2\exp(\xi_2)+k_3c_3\exp(\xi_3)+a_{123}(k_1+k_2+k_3)c_1c_2c_3\exp\{\xi_1+\xi_2+\xi_3\}}{c_1\exp(\xi_1)+c_2\exp(\xi_2)+c_3\exp(\xi_3)+a_{123}c_1c_2c_3\exp\{\xi_1+\xi_2+\xi_3\}}\right), \tag{14}$$

where, $\xi_1=k_1x+(k_1^2+k_1k_2+k_2k_3+k_1k_3)t$, $\xi_2=k_2x+(k_2^2+k_1k_2+k_2k_3+k_1k_3)t$, $\xi_3=k_3x+(k_3^2+k_1k_2+k_2k_3+k_1k_3)t$ and the corresponding potential field reads $v=-u_x$.

Set-3: $r=-1$, $a_0=a_{23}=a_{13}=a_{123}=0$, $a_{12}=const$, $w_1=k_1^2+2k_1k_2$, $w_2=k_2^2+2k_1k_2$, $w_3=k_3^2+2k_1k_2$ then

$$u(x,t)=-\left(\frac{k_1c_1\exp(\xi_1)+k_2c_2\exp(\xi_2)+k_3c_3\exp(\xi_3)+a_{12}(k_1+k_2)c_1c_2\exp\{\xi_1+\xi_2\}}{c_1\exp(\xi_1)+c_2\exp(\xi_2)+c_3\exp(\xi_3)+a_{12}c_1c_2\exp\{\xi_1+\xi_2\}}\right), \tag{15}$$

where $\xi_1=k_1x+(k_1^2+2k_1k_2)t$, $\xi_2=k_2x+(k_2^2+2k_1k_2)t$, $\xi_3=k_3x+(k_3^2+2k_1k_2)t$ and the corresponding potential field reads $v=-u_x$.

Set-4: $r=-1$, $a_0=a_{12}=a_{13}=a_{123}=0$, $a_{23}=const$, $w_1=k_1^2+2k_2k_3$, $w_2=k_2^2+2k_2k_3$, $w_3=k_3^2+2k_2k_3$ then

$$u(x,t)=-\left(\frac{k_1c_1\exp(\xi_1)+k_2c_2\exp(\xi_2)+k_3c_3\exp(\xi_3)+a_{23}(k_2+k_3)c_2c_3\exp\{\xi_2+\xi_3\}}{c_1\exp(\xi_1)+c_2\exp(\xi_2)+c_3\exp(\xi_3)+a_{23}c_2c_3\exp\{\xi_2+\xi_3\}}\right), \tag{16}$$

where $\xi_1=k_1x+(k_1^2+2k_2k_3)t$, $\xi_2=k_2x+(k_2^2+2k_2k_3)t$, $\xi_3=k_3x+(k_3^2+2k_2k_3)t$ and the corresponding potential field reads $v=-u_x$.

Set-5: $r=-1$, $a_0=a_{12}=a_{23}=a_{123}=0$, $a_{13}=const$, $w_1=k_1^2+2k_1k_3$, $w_2=k_2^2+2k_1k_3$, $w_3=k_3^2+2k_1k_3$ then

$$u(x,t)=-\left(\frac{k_1c_1\exp(\xi_1)+k_2c_2\exp(\xi_2)+k_3c_3\exp(\xi_3)+a_{13}(k_1+k_3)c_1c_3\exp\{\xi_1+\xi_3\}}{c_1\exp(\xi_1)+c_2\exp(\xi_2)+c_3\exp(\xi_3)+a_{13}c_1c_3\exp\{\xi_1+\xi_3\}}\right), \tag{17}$$

$\xi_1=k_1x+(k_1^2+2k_1k_3)t$, $\xi_2=k_2x+(k_2^2+2k_1k_3)t$, $\xi_3=k_3x+(k_3^2+2k_1k_3)t$, and the corresponding potential field reads $v=-u_x$.

Set-6: $r=-1$, $a_0=a_{12}=a_{23}=a_{13}=a_{123}=0$, $w_1=const$, $w_2=w_1+k_2^2-k_1^2$, $w_3=w_1+k_3^2-k_1^2$ then

$$u(x,t)=-\frac{k_1c_1\exp(k_1(x+w_1t))+k_2c_2\exp(k_2(x+(w_1+k_2^2-k_1^2)t))+k_3c_3\exp(k_3(x+(w_1+k_3^2-k_1^2)t))}{a_0+c_1\exp(k_1(x+k_1t))+c_2\exp(k_2(x+(w_1+k_2^2-k_1^2)t))+c_3\exp(k_{30}(x+(w_1+k_3^2-k_1^2)t))}, \tag{18}$$

where $a_0, c_1, c_2, c_3, k_1, k_2, k_3$ are arbitrary constants. The corresponding potential field reads $v=-u_x$.

Fig. 5 profile of solution Eq. (11) of Burger equation and Fig. 6 profile of solution Eq. (14) of Burger equation. We see that both Fig. 5 and Fig. 6 are elastic scattering and no fission exist, but the fusion phenomenon of the three solitary waves fusion into a one solitary wave after interaction.

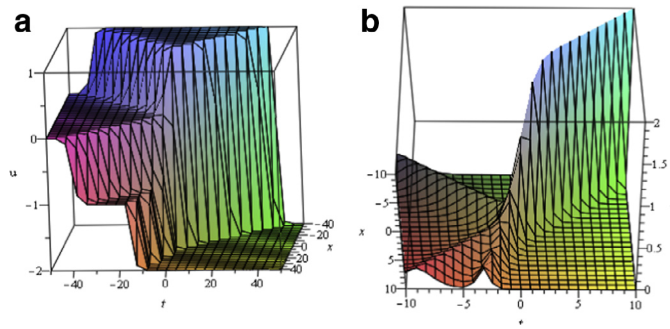

Fig. 5. (a) Profile of three solitary wave fusion solution Eq. (11) of Burger equation, (b) corresponding potential field Eq. (12) with $k_1=1, k_2=-1, k_3=2, c_1=c_3=1, c_2=2, a_0=2$.

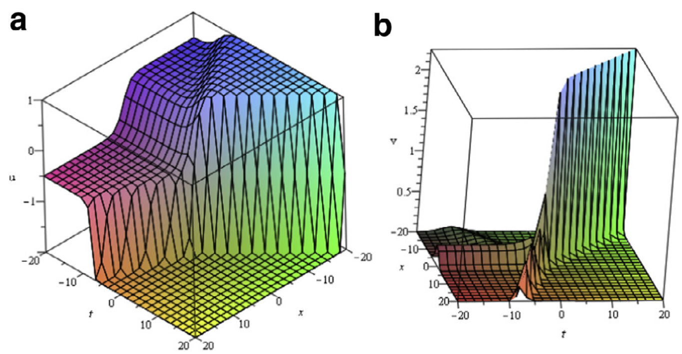

Fig. 6. (a) Profile of three solitary wave fusion solution Eq. (14) of Burger equation, (b) corresponding potential field Eq. (12) with $k_1=-1, k_2=2, k_3=-0.5, c_1=c_3=c_2=1, a_{123}=4$.

3. Multi-soliton of Sharma–Tasso–Olver equation and its fission and fusion

In this section, we bring to bear a direct rational exponential approach to explain the completely non-elastic interaction clearly of the simplest non-linear Sharma–Tasso–Olver equation [23],

$$u_t+3\alpha(u_x)^2+3\alpha u^2u_x+3\alpha uu_{xx}+\alpha u_{xxx}=0. \tag{19}$$

For single soliton solution we first consider trial solution as Eq. (2).

Inserting (2) and (19), and then maintenance the coefficients of $(\exp(k_1x+w_1t))^i$, $(i=\ldots-2,1,0,1,2\ldots)$ is zero, yields a system of algebraic equations about a_0, c_1, w_1 and $k_1$as follows:

$$2a_0c_1w_1+6\alpha ra_0k_1^3c_1-4\alpha a_0c_1k_1^3=0,$$
$$3\alpha r^2k_1^3c_1^2-3\alpha rk_1^3c_1^2+c_1^2w_1+\alpha k_1^3c_1^2=0,$$
$$w_1a_0^2+\alpha k_1^3a_0^2=0.$$

Solving this over-determined system of algebraic equations for a_0, w_1, r with the aid Maple 13, we arrive at the following solutions: $a_0=const$, $w_1=-\alpha k_1^3$, $r=1$ and thus the solution is

$$u(x,t)=\frac{k_1c_1\exp(k_1(x-\alpha k_1^2t))}{a_0+c_1\exp(k_1(x-\alpha k_1^2t))}, \tag{20}$$

and the corresponding potential field reads

$$v(x,t) = -\frac{k_1^2 c_1 \exp(k_1(x-\alpha k_1^2 t))}{a_0 + c_1 \exp(k_1(x-\alpha k_1^2 t))} + \frac{k_1^2 c_1^2 \exp(2k_1(x-\alpha k_1^2 t))}{(a_0 + c_1 \exp(k_1(x-\alpha k_1^2 t)))^2}. \tag{21}$$

Figures of Eqs. (20) and (21) are similar to Fig. 1 (for convenience we omitted these).

To achieve two soliton solutions we just suppose Eq. (5) as a trial solution. Using the same procedure like Burger equations with the help of commercial software Maple-13, we have three sets of solutions as follows:

Set-1: $r = 1,\ a_0 = const,\ a_{12} = 0,\ w_1 = -\alpha k_1^3,\ w_2 = -\alpha k_2^3$ then

$$u(x,t) = \frac{k_1 c_1 \exp(k_1(x-\alpha k_1^2 t)) + k_2 c_2 \exp(k_2(x-\alpha k_2^2 t))}{a_0 + c_1 \exp(k_1(x-\alpha k_1^2 t)) + c_2 \exp(k_2(x-\alpha k_2^2 t))}, \tag{22}$$

where $\alpha,\ a_0, c_1, c_2, k_1, k_2$ are arbitrary constants.

The corresponding potential field reads

$$v = -\frac{k_1^2 c_1 \exp(k_1(x-\alpha k_1^2 t)) + k_2^2 c_2 \exp(k_2(x-\alpha k_2^2 t))}{a_0 + c_1 \exp(k_1(x-\alpha k_1^2 t)) + c_2 \exp(k_2(x-\alpha k_2^2 t))} + \frac{\left(k_1 c_1 \exp(k_1(x-\alpha k_1^2 t)) + k_2 c_2 \exp(k_2(x-\alpha k_2^2 t))\right)^2}{\left(a_0 + c_1 \exp(k_1(x-\alpha k_1^2 t)) + c_2 \exp(k_2(x-\alpha k_2^2 t))\right)^2}. \tag{23}$$

Both fusion and fission exist in the solution Eqs. (22) and (23) of Sharma–Tasso–Olver equation but it depends on the sign of the exist parameters, from analysis our resulting phenomena are shown in the following table for the solution Eqs. (22) and (23):

α	k_1	k_2	*phenomena*
+	+	+	Fission
+	-	-	Fusion
+	-	+	Fission
+	+	-	Fusion
-	+	+	Fusion
-	-	-	Fission
-	-	+	Fusion
-	+	-	Fission

From the Fig. 7, we can see that both fission and fusion phenomena for different condition on the exists parameters for the solution Eq. (22) of STO equation.

Set-2: $r = 1,\ a_0 = 0,\ a_{12} = const,\ w_1 = -\alpha(k_1^3 + 3k_1^2 k_2 + 3k_1 k_2^2),\ w_2 = -\alpha(k_2^3 + 3k_1^2 k_2 + 3k_1 k_2^2)$ then

$$u(x,t) = \frac{k_1 c_1 \exp(\xi_1) + k_2 c_2 \exp(\xi_2) + a_{12}(k_1 + k_2) c_1 c_2 \exp\{\xi_1 + \xi_2\}}{c_1 \exp(\xi_1) + c_2 \exp(\xi_2) + a_{12} c_1 c_2 \exp\{\xi_1 + \xi_2\}}, \tag{24}$$

where $\xi_1 = k_1 x - \alpha(k_1^3 + 3k_1^2 k_2 + 3k_1 k_2^2)t,\ \xi_2 = k_2 x - \alpha(k_2^3 + 3k_1^2 k_2 + 3k_1 k_2^2)t$ and the corresponding potential field reads $v = -u_x$.

Set-3: $r = 1,\ a_0 = 0,\ a_{12} = 0,\ w_2 = const,\ w_1 = w_2 - \alpha(k_1^3 - k_2^3)$ then

$$u(x,t) = \frac{k_1 c_1 \exp(k_1 x + (w_2 - \alpha(k_1^3 - k_2^3))t) + k_2 c_2 \exp(k_1 x + w_2 t)}{c_1 \exp(k_1 x + (w_2 - \alpha(k_1^3 - k_2^3))t) + c_2 \exp(k_1 x + w_2 t)}, \tag{25}$$

and the corresponding potential field reads $v = -u_x$.

To achieve three soliton solutions, we just suppose Eq. (10) as trial solution and using the same procedure like Burger equations with the help of commercial software Maple-13, we have three sets of solutions as follows:

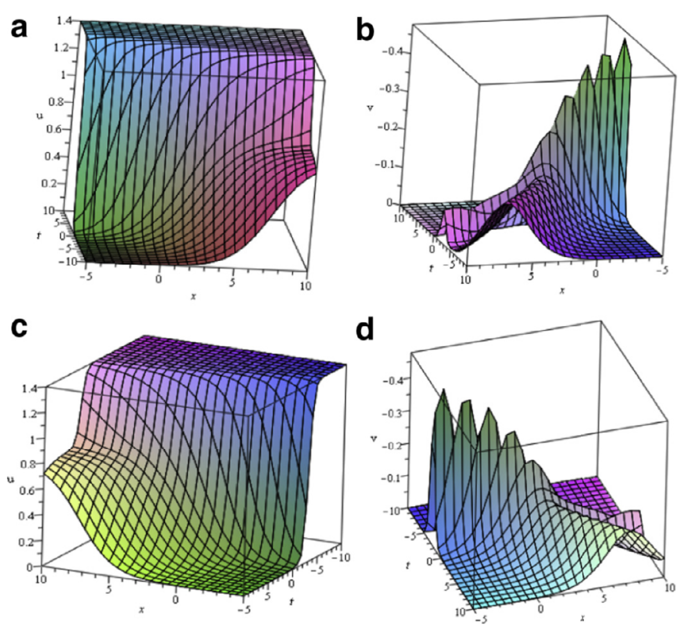

Fig. 7. (a) Profile of two solitary wave fusion solution Eq. (22) of STO equation, (b) corresponding potential field Eq. (23) with $\alpha = -1, k_1 = 0.8, k_2 = 1.4, c_1 = c_3 = c_2 = 1,\ a_0 = 2$and (c) fission of the same Eq. (22) solution, (d) corresponding potential field for $\alpha = 1, k_1 = 0.8, k_2 = 1.4, c_1 = c_3 = c_2 = 1,\ a_0 = 2$.

Set-1: $r = 1,\ a_0 = const,\ a_{12} = a_{13} = a_{23} = a_{123} = 0,\ w_1 = -\alpha k_1^3,\ w_2 = -\alpha k_2^3, w_3 = -\alpha k_3^3$ then

$$u(x,t) = \frac{k_1 c_1 \exp(k_1(x-\alpha k_1^2 t)) + k_2 c_2 \exp(k_2(x-\alpha k_2^2 t)) + k_3 c_3 \exp(k_3(x-\alpha k_3^2 t))}{a_0 + c_1 \exp(k_1(x-\alpha k_1^2 t)) + c_2 \exp(k_2(x-\alpha k_2^2 t)) + c_3 \exp(k_3(x-\alpha k_3^2 t))}, \tag{26}$$

and the corresponding potential field reads

$$v = -\frac{k_1^2 c_1 \exp(k_1(x-\alpha k_1^2 t)) + k_2^2 c_2 \exp(k_2(x-\alpha k_2^2 t)) + k_3^2 c_3 \exp(k_3(x-\alpha k_3^2 t))}{a_0 + c_1 \exp(k_1(x-\alpha k_1^2 t)) + c_2 \exp(k_2(x-\alpha k_2^2 t)) + c_3 \exp(k_3(x-\alpha k_3^2 t))} + \left(\frac{k_1 c_1 \exp(k_1(x-\alpha k_1^2 t)) + k_2 c_2 \exp(k_2(x-\alpha k_2^2 t)) + k_3 c_3 \exp(k_3(x-\alpha k_3^2 t))}{a_0 + c_1 \exp(k_1(x-\alpha k_1^2 t)) + c_2 \exp(k_2(x-\alpha k_2^2 t)) + c_3 \exp(k_3(x-\alpha k_3^2 t))}\right)^2. \tag{27}$$

Set-2: $r = 1,\ a_0 = a_{12} = a_{13} = a_{23} = a_{123} = 0,\ w_1 = w_2 + \alpha(k_2^3 - k_1^3),\ w_3 = w_2 + \alpha(k_3^3 - k_2^3),\ w_2 = const.$ or $r = 1$, $a_0 = a_{12} = a_{13} = a_{23} = a_{123} = 0,\ w_1 = w_3 + \alpha(k_3^3 - k_1^3),\ w_2 = w_3 + \alpha(k_3^3 - k_2^3),\ w_3 = const.$ then setting any one we achieve similar solution as

$$u(x,t) = \frac{k_1c_1\exp(k_1x + (w_2 + \alpha(k_2^3 - k_1^3))t) + k_2c_2\exp(k_2x + w_2t) + k_3c_3\exp(k_3x + (w_2 + \alpha(k_3^3 - k_2^3))t)}{c_1\exp(k_1x + (w_2 + \alpha(k_2^3 - k_1^3))t) + c_2\exp(k_2x + w_2t) + c_3\exp(k_3x + (w_2 + \alpha(k_3^3 - k_2^3))t)}, \tag{28}$$

and the corresponding potential field reads $v = -u_x$.

Set-3: $r = 1,\ a_0 = a_{13} = a_{23} = a_{123} = 0,\ a_{12} = const.,\ w_1 = -\alpha(k_1^3 + 3k_1^2k_2 + 3k_1k_2^2),\ w_2 = -\alpha(k_2^3 + 3k_1^2k_2 + 3k_1k_2^2),\ w_2 = -\alpha(k_3^3 + 3k_1^2k_2 + 3k_1k_2^2)$ or, $r = 1,\ a_0 = a_{12} = a_{13} = a_{123} = 0,\ a_{23} = const.,\ w_1 = -\alpha(k_1^3 + 3k_3^2k_2 + 3k_3k_2^2),\ w_2 = -\alpha(k_2^3 + 3k_3^2k_2 + 3k_3k_2^2),\ w_2 = -\alpha(k_3^3 + 3k_3^2k_2 + 3k_3k_2^2)$ or, $r = 1,\ a_0 = a_{12} = a_{23} = a_{123} = 0,\ a_{13} = const.,\ w_1 = -\alpha(k_1^3 + 3k_3^2k_1 + 3k_3k_1^2),\ w_2 = -\alpha(k_2^3 + 3k_3^2k_1 + 3k_3k_1^2),\ w_2 = -\alpha(k_3^3 + 3k_3^2k_1 + 3k_3k_1^2)$, then setting any one we achieve similar solution as

$$u(x,t) = \frac{k_1c_1\exp(\xi_1) + k_2c_2\exp(\xi_2) + k_3c_3\exp(\xi_3) + a_{12}(k_1 + k_2)c_1c_2\exp\{\xi_1 + \xi_2\}}{c_1\exp(\xi_1) + c_2\exp(\xi_2) + c_3\exp(\xi_3) + a_{12}c_1c_2\exp\{\xi_1 + \xi_2\}}, \tag{29}$$

where $\xi_1 = k_1x - \alpha(k_1^3 + 3k_1^2k_2 + 3k_1k_2^2)t,\ \xi_2 = k_2x - \alpha(k_2^3 + 3k_1^2k_2 + 3k_1k_2^2)t, \xi_3 = k_3x - \alpha(k_3^3 + 3k_1^2k_2 + 3k_1k_2^2)t$. and the corresponding potential field reads $v = -u_x$.

From the Fig. 8, we can see that both fission and fusion phenomena occurred for different condition on the exists parameters for the solution Eq. (29) of STO equation.

Set-4: $r = 1,\ a_0 = a_{12} = a_{23} = a_{13} = 0, a_{123} = const.,$ $w_1 = -\frac{\alpha}{2}(2k_1^3 + 3k_1k_2^2 + 3k_1^2k_2 + 3k_2k_3^2 + 3k_3k_2^2 + 3k_1k_3^2 + 3k_1^2k_3 + 6k_1k_2k_3),\ w_2 = -\frac{\alpha}{2}(2k_2^3 + 3k_1k_2^2 + 3k_1^2k_2 + 3k_2k_3^2 + 3k_3k_2^2 + 3k_1k_3^2 + 3k_1^2k_3 + 6k_1k_2k_3),\ w_3 = -\frac{\alpha}{2}(2k_3^3 + 3k_1k_2^2 + 3k_1^2k_2 + 3k_2k_3^2 + 3k_3k_2^2 + 3k_1k_3^2 + 3k_1^2k_3 + 6k_1k_2k_3)$then

$$u(x,t) = \frac{k_1c_1\exp(\xi_1) + k_2c_2\exp(\xi_2) + k_3c_3\exp(\xi_3) + a_{123}(k_1 + k_2 + k_3)c_1c_2c_3\exp\{\xi_1 + \xi_2 + \xi_3\}}{c_1\exp(\xi_1) + c_2\exp(\xi_2) + c_3\exp(\xi_3) + a_{123}c_1c_2c_3\exp\{\xi_1 + \xi_2 + \xi_3\}}, \tag{30}$$

where $\xi_1 = k_1x - \frac{\alpha}{2}(2k_1^3 + 3k_1k_2^2 + 3k_1^2k_2 + 3k_2k_3^2 + 3k_3k_2^2 + 3k_1k_3^2 + 3k_1^2k_3 + 6k_1k_2k_3)t,\ \xi_2 = k_2x - \frac{\alpha}{2}(2k_2^3 + 3k_1k_2^2 + 3k_1^2k_2 + 3k_2k_3^2 + 3k_3k_2^2 + 3k_1k_3^2 + 3k_1^2k_3 + 6k_1k_2k_3)t$ and $\xi_3 = k_3x - \frac{\alpha}{2}(2k_3^3 + 3k_1k_2^2 + 3k_1^2k_2 + 3k_2k_3^2 + 3k_3k_2^2 + 3k_1k_3^2 + 3k_1^2k_3 + 6k_1k_2k_3)t$, and the corresponding potential field reads $v = -u_x$.

Remark: All of the solutions available in this paper have been checked with the help of Maple-13 and we observe that they satisfy the corresponding original equation.

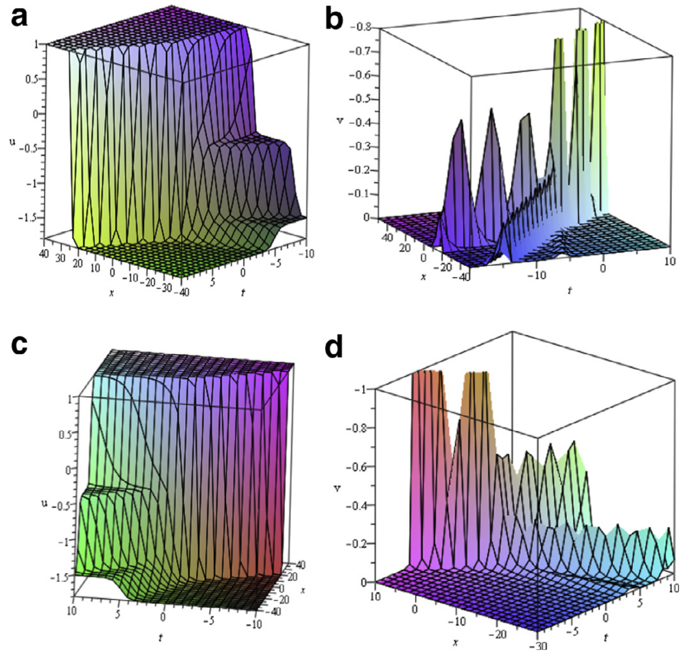

Fig. 8. (a) Profile of two solitary wave fusion solution Eq. (29) of STO equation, (b) corresponding potential field with $\alpha = 1, k_1 = 1, k_2 = -1.5, k_3 = -1.8 c_1 = c_3 = c_2 = 1,\ a_{12} = 1$and (c) fission of the same Eq. (29) solution, (d) corresponding potential field for $\alpha = -1, k_1 = 1, k_2 = -1.5, k_3 = -1.8, c_1 = c_3 = c_2 = 1,\ a_{12} = 1$.

4. Comparison

In Ref. [23] authors found only two dispersion relations and found one soliton solution for each one, two and three soliton solutions for Burger equation. When $c_1 = a_0 = 1,\ k_1 = k$, then our solution Eq. (3) reduces to solution Eq. (11) of Ref. [23], when $c_1 = c_2 = a_0 = 1$, then our solution Eq. (6) reduces to solution Eq. (14) of Ref. [23] of the Burger equation. All other solutions for Burger equation are new with both elastic and non-elastic (fusion) phenomena. When $c_1 = c_2 = a_0 = 1$, then our solution Eq. (22) reduces to solution Eq. (40) of Ref. [23] of STO equation. All other solutions for STO equation are new and non-elastic with both fusion and fission phenomena depending on the sign of the exist parameters.

5. Conclusion

We used direct rational exponential scheme to investigate fusion and fission phenomena for the Burger and STO equations. Both elastic and non-elastic fusion phenomena are found for the Burger equation. But both fusion and fission phenomena exist in the STO equation depending on the sign of the exist parameters. Some figures are provided to realize the fusion and fission phenomena of the equations. The obtained solutions may be significant and important for analyzing the nonlinear phenomena arising in engineering fields. We found fusion phenomenon of waves that increases wave height with more potential energy can be used in turbine like mechanism and found fission phenomenon of wave that divided a single wave into two or more waves that decreases wave height with smaller potential energy can be used in breaking seismic like waves. The obtain solutions can be described many physical wave phenomena habitually in diverse branches of many fields such as ocean water waves, plasma waves and ion acoustic plasma waves. In the future task, we try to find the general equation for the distribution of the energy and momentum after the soliton fission and/or fusion. Our more future task is to obtain the soliton fission and fusion solutions in higher dimensional equations.

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