



Original Article

Abundant general solitary wave solutions to the family of KdV type equations

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Abstract

This work explores the construction of more general exact traveling wave solutions of some nonlinear evolution equations (NLEEs) through the application of the $(G'/G, 1/G)$ -expansion method. This method is allied to the widely used (G'/G) -method initiated by Wang et al. and can be considered as an extension of the (G'/G) -expansion method. For effectiveness, the method is applied to the family of KdV type equations. Abundant general form solitary wave solutions as well as periodic solutions are successfully obtained through this method. Moreover, in the obtained wider set of solutions, if we set special values of the parameters, some previously known solutions are revived. The approach of this method is simple and elegantly standard. Having been computerized it is also powerful, reliable and effective.

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1. Introduction

Nonlinearity exists inherently everywhere in the real world. It remains in very near of our doorstep to any long range. Any linear or nonlinear natural phenomenon always abides by some scientific laws which are modeled by mathematical formulations or expressions. When the attribute rates of change of a quantity or time evolution event gets involved, then it can be modeled by the linear or nonlinear ordinary differential equations (ODE) or partial differential equations (PDE) or system of them. Most physical phenomena of fluid dynamics, quantum mechanics, electricity, magnetism, propagation of shallow water waves, plasma physics and many other models are governed within its domain of validity by nonlinear PDE.

To analyze the underlying mechanism or the physical effects through the model problem, it is crucial to derive the solutions of the governing equations. Explicit solution provides

important qualitative and quantitative information fundamentally in a direct way. Particularly, an analytical or closed form solution gives insight the nature of the problem clearly. As a result, research of finding exact solutions to nonlinear evolution equations is one of the most vibrant areas of mathematics, physics, chemistry, biology and so on. It has had a broad and far-reaching impact on myriad fields ranging from the mathematical to physical science.

There exists several approaches for finding exact traveling wave solutions in the literature of nonlinear problems such as the inverse scattering method [1], the Hirota's bilinear operators [2] for solving the Cauchy problem in case of integrable partial differential equations, the Backlund transformation [3] and the Wronskian determinant technique [4] and so on.

The above mentioned methods are improved by the assistance of computer software and many other algebraic methods are proposed e.g. the Jacobi elliptic function method [5], the Weierstrass function method [6], the homogeneous balance method [7], the theta function method [8], the complex hyperbolic function method [9], the exp-function method [10],

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the F-expansion method [11], the sub-ODE method [12], the symmetry method [13], the first integral method [14], the trial function method [15], the nonlinear transform method [16], the hyperbolic tangent method [17], the Adomain decomposition method [18], the sine–cosine method [19], the tanh–coth method [20] etc.

In the recent years, the (G'/G) -expansion method, introduced by Wang et al. [21] has been gained popularity to the wider area of physical science and engineering for searching different exact solutions of nonlinear evolution equations. The significance of this method is that it handles nonlinear problems by transforming into simple algebraic equations. It has been demonstrated that the technique of this method is simple and effective for seeking analytical solutions to nonlinear PDE. From that time, to improve as well as to enhance the (G'/G) -expansion method, many works have been carried out and many exact solutions have been successfully found [22,23,24]. Zhang et al. [25] improved the method to deal with nonlinear evolution equations with variable coefficients. Chao and Yu-Bin [26] modified the method to derive traveling wave solutions to the Whitham–Broer–Kaup like equations. A remarkable work was also done by Zhang [27], on some special nonlinear evolution equations where the balance numbers are not positive integers.

In this article, we have suggested and implemented the two variables $(G'/G, 1/G)$ -expansion method which is considered as the generalization of the original (G'/G) -expansion method to the celebrated and well known Korteweg de Vries (KdV)-family type equations. These equations appear in many scientific fields, for instance, model of surface waves with small amplitude and long wavelength on shallow water. The KdV equations represent the rate of change of the wave’s height in time is governed by the sum of two terms: one is nonlinear terms that have the amplitude effect and the other is dispersive terms that have effect on waves of different wavelengths to travel with different velocities.

Li et al. [28] are the pioneers of the two variables $(G'/G, 1/G)$ -expansion method and they applied it to find exact solutions to the Zakharov equation. Zayed and Abdelaziz [29] also applied this method to the Zoomeron equation and found useful solution successfully. Very recently, Demiray et al. [30] have determined the exact solutions of the Boussinesq type equations by using the $(G'/G, 1/G)$ -expansion method effectively. Yet considerable work has to be done in order to make the $(G'/G, 1/G)$ -expansion method well-set as every nonlinear PDE has its own physically significant rich shape.

The main idea of this method is that the exact traveling wave solutions of nonlinear PDEs can be expressed by a polynomial in two variables (G'/G) and $(1/G)$, where $G = G(\xi)$ satisfies a second-order linear ordinary differential equation (LODE), $G''(\xi) + \lambda G(\xi) = \mu$ in which λ and μ are constants. The degree of the polynomial can be determined by the homogeneous balance between the highest order derivatives and nonlinear terms that appear in the given PDEs. Moreover, the coefficients of this polynomial can be obtained by solving a

set of algebraic equations resulted from the process of this method.

The objective of this work is twofold: first, we present the $(G'/G, 1/G)$ -expansion method and the second is to implement the method to obtain general solitary wave solutions to the family of KdV equations.

2. Outline of the $(G'/G, 1/G)$ -expansion method

The principal ideas of the $(G'/G, 1/G)$ -expansion method are presented in the following:

First, we consider a linear ordinary differential equation (LODE) in $G = G(\xi)$ as

$$G''(\xi) + \lambda G(\xi) = \mu, \tag{2.1}$$

where λ and μ are two arbitrary constants.

Again, we consider two rational functions ϕ and ψ as

$$\phi = G'/G, \quad \psi = 1/G, \tag{2.2}$$

where G' is the derivative of G .

From Eqs. (2.1) and (2.2), the following relations can be derived easily

$$\phi' = -\phi^2 + \mu\psi - \lambda, \quad \psi' = -\phi\psi. \tag{2.3}$$

The general solutions of LODE (2.1), depend on whether the values of $\lambda(0, \lambda)0$ or $\lambda = 0$.

Case 1: When $\lambda < 0$, the general solution of LODE (2.1) is

$$G(\xi) = A_1 \sinh(\sqrt{-\lambda}\xi) + A_2 \cosh(\sqrt{-\lambda}\xi) + \frac{\mu}{\lambda}, \tag{2.4}$$

where A_1 and A_2 are two arbitrary constants.

Now, from Eqs. (2.2), (2.3) and (2.4), the following relation can be deduced $\psi^2 = \frac{-\lambda}{\lambda^2\sigma + \mu^2}(\phi^2 - 2\mu\psi + \lambda)$, where

$$\sigma = A_1^2 - A_2^2. \tag{2.5}$$

Case 2: When $\lambda > 0$, the general solution of (2.1) is the form of trigonometric function as

$$G(\xi) = A_1 \sin(\sqrt{\lambda}\xi) + A_2 \cos(\sqrt{\lambda}\xi) + \frac{\mu}{\lambda}, \tag{2.6}$$

where A_1 and A_2 are two arbitrary constants.

On the other hand, from Eqs. (2.2), (2.3) and (2.6), the following relation can be derived $\psi^2 = \frac{\lambda}{\lambda^2\rho - \mu^2}(\phi^2 - 2\mu\psi + \lambda)$ where

$$\rho = A_1^2 + A_2^2. \tag{2.7}$$

Case 3: Finally, when $\lambda = 0$, the general solution of LODE is

$$G(\xi) = \frac{\mu}{2}\xi^2 + A_1\xi + A_2 \tag{2.8}$$

and in the same aforementioned way from Eqs. (2.2), (2.3) and (2.8) the relationship between ϕ and ψ yields,

$$\psi^2 = \frac{\lambda}{A_1^2 - 2\mu A_2}(\phi^2 - 2\mu\psi), \tag{2.9}$$

where A_1 and A_2 are two arbitrary constants.

Let us now consider general nonlinear evolution equations of the form

$$F(u, u_t, u_x, u_{tt}, u_{xt}, u_{xx}, \dots) = 0, \quad (2.10)$$

where $u = u(x, t)$ is an unidentified function depends on two variables x and t . The function F is a polynomial in $u(x, t)$ and its various partial derivatives, involving nonlinear terms and the highest order derivative terms. The main procedures to find the exact solutions of this type of nonlinear evolution equation by this extension method are in the following stepwise:

Step 1: Consider the traveling wave variable

$$u(x, t) = u(\xi), \quad \xi = x - ct, \quad (2.11)$$

where c is the speed of the traveling wave. Applying the chain rule of partial derivatives and substituting the values of $u = u(x, t)$ and its various partial derivatives into Eq. (2.10), it reduces to the following ordinary differential equation (ODE):

$$F(u, -cu', u', u'', -cu'', -c^2u''', \dots) = 0, \quad (2.12)$$

where $u' = \frac{du}{d\xi}$, $u'' = \frac{d^2u}{d\xi^2}$, ... and so on.

Step 2: The ansatz of this extension method is that the solution of ODE (2.12) can be expressed by a polynomial in $\phi = (G'/G)$ and $\psi = (1/G)$ as

$$u(\xi) = \sum_{i=0}^m a_i \phi^i + \sum_{i=1}^m b_i \phi^{i-1} \psi, \quad (2.13)$$

where G satisfies the second order LODE (2.1), $a_i (i = 0, 1, 2, \dots, m)$, $b_i (i = 0, 1, 2, \dots, m)$, λ , μ and c are constants to be determined later. The degree m of the polynomial can be determined by using the homogeneous balance between the highest order derivatives and the nonlinear terms appearing in Eq. (2.12).

Step 3: Substituting the assumed solution $u = u(\xi)$ of (2.13) into Eq. (2.12), using (2.3) and (2.5), the left hand side of (2.12) becomes a polynomial in ϕ and ψ , where the degree of ψ is not larger than one. Equating each coefficient of the polynomial to zero yields a system of algebraic equations in $a_i (i = 0, 1, 2, \dots, m)$, $b_i (i = 0, 1, 2, \dots, m)$, c , $\lambda (\lambda < 0)$, μ , A_1 and A_2 .

Step 4: Solving the algebraic equations obtained in Step 3 with the help of any computer algebraic manipulating software, like Maple and substituting the values of $a_i (i = 0, 1, 2, \dots, m)$, $b_i (i = 0, 1, 2, \dots, m)$, c , $\lambda (\lambda < 0)$, μ , A_1 and A_2 into Eq. (2.13), we find the traveling wave solutions of Eq. (2.10) in terms of hyperbolic functions.

Step 5: Similarly, following the Step 3 and Step 4, substituting (2.13), into Eq. (2.12), using (2.3) and (2.7) (or (2.3) and (2.9)), we obtain the solutions of Eq. (2.10) in terms of trigonometric functions (or by rational functions).

3. Application of $(G'/G, 1/G)$ -expansion method to the family of KdV type equation

3.1. Potential KdV equation

In 1895, two Dutch physicists Diederick Korteweg and his student Gustav de Vries derived a family of nonlinear partial differential equation of the form,

$$u_t + f(u)u_x + u_{xxx} = 0, \quad (3.1.1)$$

where $u(x, t)$ is a function of space variable x and time variable t , now which is known as KdV equation in their names. The coefficients of u_x and u_{xxx} can be used as constants but these constants can be easily scaled out. This well known equation had already appeared in a work on water waves by Boussinesq in 1872. The potential KdV equation is obtained by substituting $f(u) = au_x$ in Eq. (3.1.1) by

$$u_t + a(u_x)^2 + u_{xxx} = 0. \quad (3.1.2)$$

This equation models a variety of nonlinear phenomena such as plasma waves, shallow water waves and so on. Eq. (3.1.2) shows that the rate of change of the wave's height in time is governed by the sum of two terms, one is nonlinear terms that have the amplitude effect and the other is dispersive terms that have effect on waves of different wavelengths to travel with different velocities. The derivative u_t describes the time evolution of the wave propagation in one direction. The nonlinear term uu_x characterizes the steepening of the wave, whereas the linear term u_{xxx} accounts for the spreading or dispersion of the wave. This equation is the simplest nonlinear partial differential equation embodying two effects: nonlinearity represented by uu_x and linear dispersion represented by u_{xxx} . The fragile balance between the weak nonlinearity and the linear dispersion defines the formulation of soliton type traveling wave that consists of single humped wave.

Now, we apply the introduced method to obtain new and more general exact traveling wave solutions of the potential KdV Eq. (3.1.2).

By using the wave transformation $\xi = x - ct$, Eq. (3.1.2) converts into a nonlinear ODE in single independent variable ξ as

$$-cu' + a(u')^2 + u''' = 0, \quad (3.1.3)$$

where $u' = \frac{du}{d\xi}$, $u''' = \frac{d^3u}{d\xi^3}$.

Considering the homogeneous balance between the nonlinear term $(u')^2$ and the highest order derivative u''' , it yields $m = 1$. On substituting the value of m in Eq. (2.13) our ansatz solution is of the form as

$$u(\xi) = a_0 + a_1\phi + b_1\psi, \quad (3.1.4)$$

where the coefficients a_0, a_1, b_1 are to be determined.

Case 1: When $\lambda < 0$ (Hyperbolic function solutions)

Substituting Eq. (3.1.4), along with Eqs. (2.3) and (2.5) into Eq. (3.1.3), the left hand side of (3.1.3) becomes a polynomial in ϕ and ψ . Setting each coefficient of the polynomial equal to zero, we derive a

set of algebraic equations for $c, a, a_0, a_1, b_1, \lambda$ and μ , as follows:

$$\phi^4 : a\lambda^4\sigma^2a_1^2 + 2a\lambda^2\mu^2\sigma a_1^2 - a\lambda^3\sigma b_1^2 + a\mu^4a_1^2 - 6\lambda^4\sigma^2a_1 - a\lambda\mu^2b_1^2 - 12\lambda^2\mu^2\sigma a_1$$

$$\phi^3\psi : 2a\lambda^4\sigma^2a_1 + 4a\lambda^2\mu^2\sigma a_1b_1 + 2a\mu^4a_1b_1 - 6\lambda^4\sigma^2b_1 - 12\lambda^2\mu^2\sigma b_1 - 6\mu^4b$$

$$\phi^3 : 2a\lambda^3\mu\sigma a_1b_1 + 2a\lambda\mu^3a_1b_1 - 6\lambda^3\mu\sigma b_1 - 6\lambda\mu^3b_1$$

$$\phi^2\psi : -2a\lambda^4\mu\sigma^2a_1^2 - 4a\lambda^2\mu^3\sigma a_1^2 + 2a\lambda^3\mu\sigma b_1^2 - 2a\mu^5a_1^2 - 12\lambda^4\mu\sigma^2a_1 + 2a\lambda\mu^3b_1^2 + 24\lambda^2\mu^3\sigma a_1 + 12\mu^5a_1$$

$$\phi^2 : 2a\lambda^5\sigma^2a_1^2 + 3a\lambda^3\mu^2\sigma a_1^2 - a\lambda^4\sigma b_1^2 + a\lambda\mu^4a_1^2 + c\lambda^4\sigma^2a_1 - 8\lambda^5\sigma^2a_1 - a\lambda^2\mu^2b_1^2 + 2c\lambda^2\mu^2\sigma a_1 - 13\lambda^3\mu^2\sigma a_1 + c\mu^4a_1 - 5\lambda\mu^4a_1.$$

$$\phi\psi : 2a\lambda^5\sigma^2a_1b_1 - 2a\lambda\mu^4a_1b_1 + c\lambda^4\sigma^2b - 5\lambda^5\sigma^2b_1 + 2c\lambda^2\mu^2\sigma b_1 + 2\lambda^3\mu^2\sigma b_1 + c\mu^4b_1 + 7\lambda\mu^4b_1$$

$$\phi : 2a\lambda^4\mu\sigma a_1b_1 + 2a\lambda^2\mu^3a_1b_1 - 6\lambda^4\mu\sigma b_1 - 6\lambda^2\mu^3b_1$$

$$\psi : -2a\lambda^5\mu\sigma^2a_1^2 - 2a\lambda^3\mu^3\sigma a_1^2 - c\lambda^4\mu\sigma^2a_1 + 5\lambda^5\mu\sigma^2a_1 - 2c\lambda^2\mu^3\sigma a_1 + 4\lambda^3\mu^3\sigma a_1 - c\mu^5a_1 - \lambda\mu^5a_1$$

$$\phi^0 : a\lambda^6\sigma^2a_1^2 + a\lambda^4\mu^2\sigma a_1^2 + c\lambda^5\sigma^2a_1 - 2\lambda^6\sigma^2a_1 + 2c\lambda^3\mu^2\sigma a_1 - \lambda^4\mu^2\sigma a_1 + c\lambda\mu^4a_1 + \lambda^2\mu^4a_1$$

Solving the above algebraic equations by using Maple, we get the following results:

$$a_0 = a_0, a_1 = \frac{3}{a}, b_1 = \frac{3\sqrt{-(\lambda^2\sigma + \mu^2)}}{a\sqrt{\lambda}}, c = -\lambda. \tag{3.1.5}$$

Therefore, substituting the above values in Eq. (3.1.4), we get

$$u(\xi) = \frac{3\sqrt{-\lambda}}{a} \frac{(A_1 \cosh(\sqrt{-\lambda}\xi) + A_2 \sinh(\sqrt{-\lambda}\xi))}{(A_1 \sinh(\sqrt{-\lambda}\xi) + A_2 \cosh(\sqrt{-\lambda}\xi) + \frac{\mu}{\lambda})} + \frac{3}{a\sqrt{\lambda}} \frac{\sqrt{-(\lambda^2\sigma + \mu^2)}}{(A_1 \sinh(\sqrt{-\lambda}\xi) + A_2 \cosh(\sqrt{-\lambda}\xi) + \frac{\mu}{\lambda})} + a_0 \tag{3.1.6}$$

In special cases, if taking $A_1 = 0, A_2 > 0$ and $\mu = 0$, Eq. (3.1.6) becomes

$$u(\xi) = \frac{3\sqrt{c}}{a} (\tanh(\sqrt{c}\xi) + \operatorname{sech}(\sqrt{c}\xi)) + a_0$$

Substituting $\xi = x - ct$ in the above equation, we get the solution of the potential KdV Eq. (3.1.2) as

$$u_1(x, t) = \frac{3\sqrt{c}}{a} (\tanh(\sqrt{c}(x - ct)) + \operatorname{sech}(\sqrt{c}(x - ct))) + a_0. \tag{3.1.7}$$

In particular, if taking $A_1 > 0, A_2 = 0$ and $\mu = 0$, then the solution of the potential KdV equation becomes

$$u_2(x, t) = \frac{3\sqrt{c}}{a} (\coth(\sqrt{c}(x - ct)) + \operatorname{csch}(\sqrt{c}(x - ct))) + a_0. \tag{3.1.8}$$

Case 2: When $\lambda > 0$ (Trigonometric function solutions)

In the same way, as stated in case 1, substituting Eq. (3.1.4), along with Eqs. (2.3) and (2.7) into Eq. (3.1.3), the left hand side of (3.1.3) becomes a polynomial in ϕ and ψ . Again, setting each coefficient of this polynomial to zero, we find a set of algebraic equations for $c, a, a_0, a_1, b_1, \lambda$ and μ . For minimalism, the equations are not given herein. Solving these algebraic equations by Maple, the following values are obtained:

$$a_0 = a_0, a_1 = \frac{3}{a}, b_1 = \frac{3\sqrt{(\lambda^2\rho - \mu^2)}}{a\sqrt{\lambda}}, c = -\lambda. \tag{3.1.9}$$

Substituting the above values in Eq. (3.1.4), we get the solution of the potential KdV Eq. (3.1.2) as

$$u(\xi) = \frac{3\sqrt{\lambda}}{a} \frac{(A_1 \cos(\sqrt{\lambda}\xi) - A_2 \sin(\sqrt{\lambda}\xi))}{(A_1 \sin(\sqrt{\lambda}\xi) + A_2 \cos(\sqrt{\lambda}\xi) + \frac{\mu}{\lambda})} + \frac{3}{a\sqrt{\lambda}} \frac{\sqrt{(\lambda^2\rho - \mu^2)}}{(A_1 \sin(\sqrt{\lambda}\xi) + A_2 \cos(\sqrt{\lambda}\xi) + \frac{\mu}{\lambda})} + a_0 \tag{3.1.10}$$

In particular, when $A_1 = 0, A_2 > 0$ and $\mu = 0$, solution (3.1.10) simplified as

$$u(\xi) = \frac{-3\sqrt{-c}}{a} (\tan(\sqrt{-c}\xi) + \sec(\sqrt{-c}\xi)) + a_0, c > 0.$$

On substituting $\xi = x - ct$, we obtain

$$u_3(x, t) = \frac{3\sqrt{c}}{a}(\tan(\sqrt{-c}(x - ct)) + \sec(\sqrt{-c}(x - ct))) + a_0, \quad c > 0. \tag{3.1.11}$$

In special cases, when $A_1 > 0, A_2 = 0$ and $\mu = 0$, then (3.1.10) can be written as

$$u_4(x, t) = \frac{-3\sqrt{-c}}{a}(\cot(\sqrt{-c}(x - ct)) + \csc(\sqrt{-c}(x - ct))) + a_0, \quad c > 0. \tag{3.1.12}$$

Case 3: When $\lambda = 0$ (Rational function solution)

Similarly, as stated in case 1 or case 2, substituting Eq. (3.1.14), along with Eqs. (2.3) and (2.9) into Eq. (3.1.13), the left hand side of (3.1.3) becomes a polynomial in ϕ and ψ . For simplicity and concise, the algebraic equations are not given herein. Solving these algebraic equations by using Maple, the following values are resulted:

$$a_0 = a_0, a_1 = \frac{3}{a}, b_1 = 0, \mu = \frac{1}{2} \frac{A_1^2}{A_2}. \tag{3.1.13}$$

Substituting the above values in Eq. (3.1.4), we get the solution of the potential KdV equation as

$$u_5(x, t) = \frac{\left(\frac{1}{2} \frac{A_1^2}{A_2}(x - ct) + A_1\right)a_1}{\frac{1}{4} \frac{A_1^2}{A_2}(x - ct)^2 + A_1(x - ct) + A_2} + a_0, \tag{3.1.14}$$

where A_1 and A_2 are two arbitrary constants.

3.2. Complex modified KdV (CMKdV) equation

The complex modified KdV equation is of the form

$$w_t + w_{xxx} + \alpha(|w|^2 w)_x = 0, \tag{3.2.1}$$

where w is a complex valued function of the spatial coordinate x and the time variable t , α is a real parameter. This equation models the nonlinear evolution plasma waves as well as incorporates the propagation of transverse waves in a molecular chain model with a general elastic solid [31]. For analysis, to decompose w of Eq. (3.2.1), into real and imaginary parts, setting $w = u + iv, i^2 = -1$, it yields the coupled pair of the modified (mKdV) equations as

$$\begin{aligned} u_t + u_{xxx} + \alpha[(u^2 + v^2)u]_x &= 0 \\ v_t + v_{xxx} + \alpha[(u^2 + v^2)v]_x &= 0 \end{aligned} \tag{3.2.2}$$

Using the wave variable $\xi = x - ct$ into the system of Eq. (3.2.2), and integrating we get

$$\begin{aligned} -cu + \alpha u^3 + \alpha uv^2 + u'' &= 0 \\ -cv + \alpha v^3 + \alpha vu^2 + v'' &= 0 \end{aligned} \tag{3.2.3}$$

As discussed in Section 2, suppose that the solutions of the above system in polynomials are

$$\begin{aligned} u(\xi) &= \sum_{i=0}^m a_i \phi^i + \sum_{i=1}^{m_1} b_i \phi^{i-1} \psi \\ v(\xi) &= \sum_{i=0}^m c_i \phi^i + \sum_{i=1}^{m_2} d_i \phi^{i-1} \psi \end{aligned} \tag{3.2.4}$$

where the coefficients $a_i, b_i (i = 0, 1, 2, \dots, m_1)$, and $c_i, d_i (i = 0, 1, 2, \dots, m_2)$, are to be determined. By balancing the higher order derivatives and nonlinear terms in Eq. (3.2.3), we get

$$m_1 = m_2 = 1 \tag{3.2.1}$$

Thus, solution formula Eq. (3.2.4) becomes

$$\begin{aligned} u(\xi) &= a_0 + a_1 \phi + b_1 \psi \\ v(\xi) &= c_0 + c_1 \phi + d_1 \psi. \end{aligned} \tag{3.2.6}$$

Case 1: When $\lambda < 0$ (Hyperbolic function solutions)

Substituting the value of $u(\xi)$ and $v(\xi)$ from Eq. (3.2.6), into Eq. (3.2.3), in conjunction with (2.3) and (2.5), the left hand side of (3.2.3) becomes a polynomial in ϕ and ψ . Equating each coefficient of this polynomial to zero and solving with the help of Maple we get, (for simplicity and concise the equations are not given)

$$\begin{aligned} a_0 = a_0, a_1 = \frac{1}{\sqrt{-2\alpha}}, b_1 = \frac{3\sqrt{(\lambda^2\sigma + \mu^2)}}{\sqrt{2\alpha\lambda}}, c = \alpha v^2 + \frac{\lambda}{2} \\ a_0 = a_0, a_1 = \frac{1}{\sqrt{-2\alpha}}, b_1 = \frac{3\sqrt{(\lambda^2\sigma + \mu^2)}}{\sqrt{2\alpha\lambda}}, c = \alpha u^2 + \frac{\lambda}{2} \end{aligned} \tag{3.2.7}$$

Therefore, substituting the above values in Eq. (3.2.6), we get the solution of Eq. (3.2.2) as

$$\begin{aligned} u(\xi) &= v(\xi) \\ &= \sqrt{\frac{\lambda}{2\alpha}} \frac{(A_1 \cosh(\sqrt{-\lambda}\xi) + A_2 \sinh(\sqrt{-\lambda}\xi))}{(A_1 \sinh(\sqrt{-\lambda}\xi) + A_2 \cosh(\sqrt{-\lambda}\xi) + \frac{\mu}{\lambda})} \\ &\quad + \frac{1}{\sqrt{2\alpha\lambda}} \frac{\sqrt{(\lambda^2\sigma + \mu^2)}}{(A_1 \sinh(\sqrt{-\lambda}\xi) + A_2 \cosh(\sqrt{-\lambda}\xi) + \frac{\mu}{\lambda})} \end{aligned} \tag{3.2.8}$$

Now, for particular cases, if we set $A_1 = 0, A_2 > 0$ and $\mu = 0$, it implies that

$$\begin{aligned} u(x, t) &= v(x, t) \\ &= \sqrt{\frac{\lambda}{2\alpha}} [\tanh(\sqrt{-\lambda}(x - ct)) + \operatorname{sech}\sqrt{-\lambda}(x - ct)]. \end{aligned}$$

Noting that, $w(x, t) = u(x, t) + iv(x, t)$, the solution of the CMKdV Eq. (3.2.1) is

$$\begin{aligned} w_1(x, t) &= (1 + i) \sqrt{\frac{\lambda}{2\alpha}} \\ &\quad \times [\tanh(\sqrt{-\lambda}(x - ct)) + \operatorname{sech}\sqrt{-\lambda}(x - ct)]. \end{aligned} \tag{3.2.9}$$

Again, for special cases, if taking $A_1 > 0$, $A_2 = 0$ and $\mu = 0$, implies that

$$u(x, t) = v(x, t) = \sqrt{\frac{\lambda}{2\alpha}} [\coth(\sqrt{-\lambda}(x - ct) + \operatorname{csch}\sqrt{-\lambda}(x - ct))].$$

Therefore, the solution of CMKdV Eq. (3.2.1) is

$$w_2(x, t) = (1 + i) \sqrt{\frac{\lambda}{2\alpha}} \times [\coth(\sqrt{-\lambda}(x - ct) + \operatorname{csch}\sqrt{-\lambda}(x - ct))]. \quad (3.2.10)$$

Case 2: When $\lambda > 0$ (Trigonometric function solutions)

Again, substituting the value of $u(\xi)$ and $v(\xi)$ of Eq. (3.2.6), into Eq. (3.2.3), together with (2.3) and (2.7), the left hand side of Eq. (3.2.3) transforms into a polynomial in ϕ and ψ . Equating each coefficient of this polynomial to zero and solving with the help of Maple we get,

$$u(\xi) = v(\xi) = \sqrt{\frac{\lambda}{-2\alpha}} \frac{(A_1 \cos(\sqrt{\lambda}\xi) - A_2 \sin(\sqrt{\lambda}\xi))}{(A_1 \sin(\sqrt{\lambda}\xi) + A_2 \cos(\sqrt{\lambda}\xi) + \frac{\mu}{\lambda})} + \frac{1}{\sqrt{2\alpha\lambda}} \frac{\sqrt{(-\lambda^2\rho + \mu^2)}}{(A_1 \sin(\sqrt{\lambda}\xi) + A_2 \cos(\sqrt{\lambda}\xi) + \frac{\mu}{\lambda})}. \quad (3.2.11)$$

In particular, if taking $A_1 = 0$, $A_2 > 0$ and $\mu = 0$, it yields

$$u(x, t) = v(x, t) = \sqrt{\frac{\lambda}{-2\alpha}} [-\tan(\sqrt{\lambda}(x - ct) + \sec\sqrt{\lambda}(x - ct))].$$

Therefore, the solution of Eq. (3.2.1) is

$$w_3(x, t) = (1 + i) \sqrt{\frac{\lambda}{-2\alpha}} \times [-\tan(\sqrt{\lambda}(x - ct) + \sec\sqrt{\lambda}(x - ct))]. \quad (3.2.12)$$

Again for special cases, when $A_1 > 0$, $A_2 = 0$ and $\mu = 0$, it yields

$$u(x, t) = v(x, t) = \sqrt{\frac{\lambda}{-2\alpha}} [\cot(\sqrt{\lambda}(x - ct) + \operatorname{csc}\sqrt{\lambda}(x - ct))].$$

Thus, the solution of the Eq. (3.2.1) is

$$w_4(x, t) = (1 + i) \sqrt{\frac{\lambda}{-2\alpha}} [\cot(\sqrt{\lambda}(x - ct) + \operatorname{csc}\sqrt{\lambda}(x - ct))]. \quad (3.2.13)$$

Case 3: When $\lambda = 0$ (Rational function solution)

In a similar fashion, as stated in *case 1*, substituting the value of $u(\xi)$ and $v(\xi)$ of Eq. (3.2.6), into Eq. (3.2.3), alongside with (2.3) and (2.9), the left hand side of Eq. (3.2.3) becomes a polynomial in ϕ and ψ . For brevity and concise, the

algebraic equations are not given herein. Solving these algebraic equations using with Maple, the following values are resulted:

$$\mu = 0, a_0 = 0, a_1 = \sqrt{-\frac{2}{\alpha}}, b_1 = 0, c = \alpha v^2 + 2\lambda. \quad (3.2.14)$$

On substituting these values in (3.2.6), we get the solution of Eq. (3.2.2) is

$$u(x, t) = v(x, t) = \frac{A_1 \sqrt{-\frac{2}{\alpha}}}{A_1(x - ct) + A_2},$$

where A_1 and A_2 are two arbitrary constants.

Therefore the solution of the complex modified KdV Eq. (3.2.1), is

$$w_5(x, t) = (1 + i) \frac{A_1 \sqrt{-\frac{2}{\alpha}}}{A_1(x - ct) + A_2}. \quad (3.2.15)$$

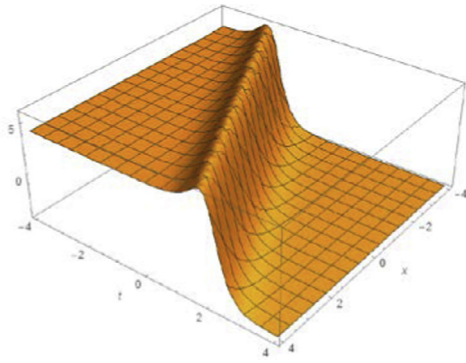
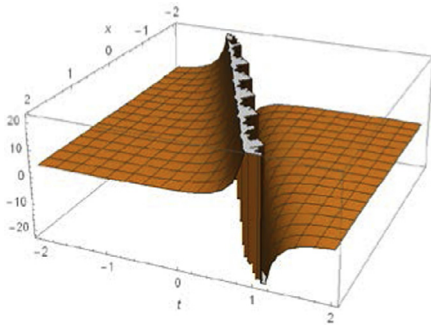
The other different types of solution of these equations are carried out by different authors [31–34], and it is worthy to note that the solutions we obtained in here are different.

4. Results and discussion

The key idea of $(G'/G, 1/G)$ -expansion method is to present the solution of a NLEE by a polynomial in two variables (G'/G) and $(1/G)$, where $G = G(\xi)$ satisfies an auxiliary second-order LODE whereas Li et al. [28], considered the solution of the NLEE as single variable function of (G'/G) and the auxiliary equation is different from the presented approach. In special case, when $\mu = 0$ in Eq. (2.1) and $b_i = 0$ in Eq. (2.13), the $(G'/G, 1/G)$ -expansion method reduces to the original (G'/G) -expansion method. As a result, the $(G'/G, 1/G)$ -expansion method can be regarded as a generalization of the (G'/G) -expansion method.

Wazwaz [31] derived exact solutions of the family of KdV type equations. Most of the solutions of Wazwaz, (A.1)–(A.8), are in the form hyperbolic or trigonometric functions individually (see Appendix). Herein, our derived solutions $u_1(x, t) - u_5(x, t)$ and $w_1(x, t) - w_5(x, t)$ are quite different, they are in the form of linear combination of hyperbolic or trigonometric functions or in the form of rational function. Thus, these solutions are new.

Besides Wazwaz solutions, other notable three, among many remarkable works for exact solutions of the KdV type equations have been found in [32,33,34]. Huiqun [32], applied an approach for finding exact solution of the KdV equation and obtained solutions (A.9)–(A.11) which is different from our approach and the obtained solutions are also different from his work. Gang-Wei et al. [33] applied the (G'/G) -expansion method and the exp-function method to the potential KdV equation for the singular soliton solution and all the solutions derived in this work are completely different from the solutions obtained in this article. Kumar and Chand [34] found bright and dark soliton solutions of the complex modified KdV equation by a new approach which is also entirely different from the present work.

Fig. 1. Graph of soliton type of traveling wave of $u_1(x, t)$.Fig. 2. Graph of singular traveling wave of $u_2(x, t)$.

Some of our obtained traveling wave solutions are graphically presented in the following figures:

Figs. 1 and 2 are drawn for the particular value of $a = 1$, $c = 2$ and in the range of $-4 \leq x, t \leq 4$.

Numerous traditional methods exist in finding the analytical solution of NLEEs and each of them some advantages and disadvantages. Many methods provide solutions in the form of series and raise a burning issue to investigate the convergence of approximation series e.g. Adomian decomposition method [18,35], depends only on the initial conditions and may need to test the convergence of the obtained solution. Some methods need linearization or to convert the inhomogeneous boundary conditions to homogeneous and so on. In addition, all numerical methods e.g. finite difference or finite element methods, it is necessary to have boundary and initial conditions. The main advantage of the $(G'/G, 1/G)$ -expansion method over other methods is that it attacks the problems in a straightforward fashion without using linearization, perturbation or any other restrictive assumption that may change the physical behavior of the model under discussion. Moreover, the availability of computational software like Maple or Mathematica facilitates the tedious algebraic calculations.

5. Conclusion

In this article, we have suggested and applied the $(G'/G, 1/G)$ -expansion method to obtain closed form traveling wave solutions of two nonlinear evolution equations. Typically, the family of KdV-type equation is

considered herein to illustrate the effectiveness of the method and some newborn solutions are successfully obtained. The derived traveling wave solutions are expressed in terms of hyperbolic, trigonometric and rational functions involving some free parameters. The procedure of this method is quite simple and the computational techniques are straightforward, efficient, as well as, practically well suited for handling nonlinear evolution equations. The suggested method in this article is more effective and general than that of the other methods as it gives exact solutions in more general forms. The new type of solutions obtained in this work might have significant impact on future researches. It will be worthy for further studies in physical sciences.

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Appendix

Wazwaz solution [31]

Wazwaz [31] investigated exact solutions of the KdV type equations by sine–cosine and tanh–coth method. He derived four explicit solutions as follows:

$$u_1(x, t) = \frac{3\sqrt{c}}{a} \tanh \left[\frac{\sqrt{c}}{2}(x - ct) \right], \quad c > 0 \quad (\text{A.1})$$

$$u_2(x, t) = \frac{3\sqrt{c}}{a} \coth \left[\frac{\sqrt{c}}{2}(x - ct) \right], \quad c > 0 \quad (\text{A.2})$$

$$u_3(x, t) = \frac{1}{\sqrt{\frac{3cR^2 - aR\sqrt{c}}{3c} + R \tanh \left[\sqrt{\frac{c}{2}}(x - ct) \right]}} \quad (\text{A.3})$$

$$u_4(x, t) = \frac{1}{\sqrt{\frac{3cR^2 - aR\sqrt{c}}{3c} + R \coth \left[\sqrt{\frac{c}{2}}(x - ct) \right]}} \quad (\text{A.4})$$

For the case of complex modified KdV equation, four solutions are obtained by sine–cosine method and twelve solutions are obtained by tanh–coth method. Notice that among sixteen solutions some are identical with others or some are combination of others. Here few solutions are given:

$$w_1(x, t) = (1 + i) \sqrt{\frac{c}{\alpha}} \csc(\sqrt{-c}(x - ct)), \quad 0 < \mu(x - ct) < \pi \quad (\text{A.5})$$

$$w_2(x, t) = (1 + i) \sqrt{\frac{c}{\alpha}} \sec(\sqrt{-c}(x - ct)), \quad 0 < \mu(x - ct) < \pi \quad (\text{A.6})$$

$$w_3(x, t) = i \frac{1}{2} \sqrt{\frac{c}{\alpha}} \left(\tanh \left(\frac{1}{2} \sqrt{\frac{-c}{2}} (x - ct) \right) \right) + \coth \left(\frac{1}{2} \sqrt{\frac{-c}{2}} (x - ct) \right) \quad (\text{A.7})$$

$$w_4(x, t) = i \frac{1}{2} \sqrt{\frac{-c}{2\alpha}} \left(\tanh \left(\frac{1}{2} \sqrt{c} (x - ct) \right) \right) + \coth \left(\frac{1}{2} \sqrt{c} (x - ct) \right) \quad (\text{A.8})$$

Huiqun's solution [32]

Huiqun [32] derived solutions of the complex KdV equation, some of them are followings:

$$w_5(x, t) = 2k^2b \tanh^2(\sqrt{-bz}) - \frac{1}{3}(2k^2b + c) + i \left(\pm \sqrt{-\frac{4bk^2}{3}(k^2b - c)} \tanh(\sqrt{-bz}) + b_0 \right), \quad z = ik(x - ct) \quad (\text{A.9})$$

$$w_6(x, t) = -2k^2b \tanh^2(\sqrt{bz}) - \frac{1}{3}(2k^2b + c) + i \left(\pm \sqrt{\frac{4bk^2}{3}(k^2b - c)} \tan(\sqrt{bz}) + b_0 \right), \quad z = ik(x - ct) \quad (\text{A.10})$$

$$w_7(x, t) = \pm \sqrt{\frac{2k^2b}{\alpha}} [\tan(\sqrt{bz}) \pm \cot(\sqrt{bz})], \quad z = ik(x - ct) \quad (\text{A.11})$$

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