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A new integrable nonlocal modified KdV equation: Abundant solutions with distinct physical structures

Abdul-Majid Wazwaz

Department of Mathematics, Saint Xavier University, Chicago, IL 60655, United States

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Abstract

In this work we study a new integrable nonlocal modified Korteweg–de Vries equation (mKdV) which arises from a reduction of the AKNS scattering problem. We use a variety of distinct techniques to determine abundant solutions with distinct physical structures. We show that this nonlocal equation possesses a family of traveling solitary wave solutions that include solitons, kinks, periodic and singular solutions.

Keywords: Nonlocal modified KdV equation; Soliton solutions; Periodic solutions; Exponential solutions.

1. Introduction

Ablowitz and Musslimani [1] introduced a new integrable nonlocal nonlinear Schrödinger equation given as

$$iq_t(x,t) = q_{xx} \pm 2q^2(x,t)q^*(-x,t), \tag{1}$$

where * denotes complex conjugate. The nonlocality occurs where one of the nonlinear terms, namely $q^*(-x,t)$ has the dependent variable evaluated at $-x$ instead of x [1]. A detailed comparison between the nonlocal Schrödinger equation with the classical Schrödinger equation was given in [1]. The nonlocal nonlinear Schrödinger equation (1) describes wave propagation in nonlinear PT symmetric media.

Recently, a new integrable nonlocal modified Korteweg–de Vries equation was given in [1,2] which reads

$$u_t(x,t) + 6u(x,t)u(-x,-t)u_x(x,t) + u_{xxx}(x,t) = 0, \tag{2}$$

which arises from a reduction of the AKNS scattering problem. This in turn shows that Eq. (2) is Lax integrable [1,2]. The nonlocality shows up as sign inversions [1] in both x and t. Recall that a nonlocal equation is a relation for which the opposite happens. When $u(-x,-t) = u(x,t)$, then Eq. (2) reduces to its standard counterparts mKdV [1], which is given by

$$u_t(x,t) + 6u^2(x,t)u_x(x,t) + u_{xxx}(x,t) = 0. \tag{3}$$

It is interesting to note that the KdV, mKdV, and Schrödinger equations are all integrable. Among the intriguing and relevant features of these equations are the multiple soliton solutions and infinite number of conserved quantities they possess. In [1,2], it was shown that the nonlocal Schrödinger (NLS) equation and the nonlocal modified KdV equation possess new properties which are different from the ones of classical equations. In [1], a detailed study of the inverse scattering transform of the nonlocal Schrödinger equation was carried out, where explicit time-periodic soliton solutions were obtained.

In [2], the Darboux transformation for the nonlocal mKdV equation was constructed, and different kinds of exact solutions including soliton, kink, antikink, complexiton, rogue-wave solution, and nonlocalized solution with singularities were obtained. The findings in [1,2] have stimulated research work in nonlocal differential equations.

Nonlinear phenomena appear in a wide variety of scientific applications such as plasma physics, quantum physics, solid-state physics, optical fibers, chemistry, biology and fluid dynamics. The determination of exact solutions can greatly facilitate the features of the examined equations. Various

E-mail address: wazwaz@sxu.edu.

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methods [3–20] have been used to handle nonlinear equations. Examples of the methods that have been used are the Hirota bilinear method, the Bäcklund transformation method, Darboux transformation, Pfaffian technique, the inverse scattering method, the Painlevé analysis, the generalized symmetry method, the subsidiary ordinary differential equation method (sub-ODE for short), the coupled amplitude-phase formulation, sine-cosine method, sech-tanh method, the mapping and the deformation approach, and many other methods.

In this work, the equation that we will be studying is the nonlocal modified KdV equation (2). We aim to use a variety of distinct techniques aiming to formally derive new traveling wave solutions, such as solitons, kinks, periodic, and singular.

2. A soliton solution

To determine a soliton solution for the nonlocal mKdV (2), we substitute

$$u(x,t) = e^{\theta} = e^{kx-ct}, \tag{4}$$

into the linear terms of (2) to find the dispersion relation as

$$c = k^3. \tag{5}$$

Consequently, the dispersion variable becomes

$$\theta = kx - k^3 t. \tag{6}$$

We next use the transformation

$$u(x,t) = R(\arctan(\theta))_x, \tag{7}$$

where R is a constant that will be determined. Substituting (7) in (2), and solving for R, we find that single soliton solution exists only if

$$R = \pm 2. \tag{8}$$

This in turn gives the following single soliton solution

$$u(x,t) = \pm \frac{2k\, e^{kx-k^3 t}}{1+e^{2kx-2k^3 t}}. \tag{9}$$

3. A kink solution

Proceeding as before, we find the dispersion relation as

$$c = k^3. \tag{10}$$

Consequently, the dispersion variable becomes

$$\theta = kx - k^3 t. \tag{11}$$

We next use the transformation

$$u(x,t) = R(\ln(1+\theta))_x, \tag{12}$$

where R is a constant that will be determined. Substituting (12) in (2), and solving for R, we find that single kink solution exists only if

$$R = \pm 1, \tag{13}$$

which gives the following single kink solution

$$u(x,t) = \pm \frac{k\, e^{kx-k^3 t}}{1+e^{kx-k^3 t}}. \tag{14}$$

It is interesting to know that the standard mKdV equation does not give kink solutions. This confirms the conclusions made in [1,2] that the nonlocal mKdV equation gives different solutions compared to the classical mKdV equation.

4. Other kink and singular solutions

To derive a variety of other kink and singular solutions, we first assume that the solution takes the form

$$u(x,t) = \frac{1}{a_0 + a_1 e^{kx-ct}}, \tag{15}$$

where a_0 and a_1 are constants to be determined. Substituting this form in (2), and solving resulting equation we obtain the following set of solutions for a_0 and a_1:

$$\begin{aligned} &a_0 = \pm\frac{1}{k}, a_1 = \pm\frac{1}{k}, c = k^3, \\ &a_0 = \mp\frac{1}{k}, a_1 = \pm\frac{1}{k}, c = k^3, \end{aligned} \tag{16}$$

which gives the following kink solutions

$$u(x,t) = \pm \frac{k}{1+e^{kx-k^3 t}}, \tag{17}$$

and the singular solutions

$$u(x,t) = \mp \frac{k}{1-e^{kx-k^3 t}}, \tag{18}$$

respectively.

However, the obtained result in this section can be generalized to the following assumption

$$u(x,t) = \frac{1}{a_0 + a_1 e^{nkx-nct}}. \tag{19}$$

Proceeding as before, we obtain the following set for a_0 and a_1:

$$\begin{aligned} &a_0 = \pm\frac{1}{nk}, a_1 = \pm\frac{1}{nk}, c = n^3k^3, \\ &a_0 = \mp\frac{1}{nk}, a_1 = \pm\frac{1}{nk}, c = n^3k^3. \end{aligned} \tag{20}$$

Consequently, we obtain the generalized kink solutions

$$u(x,t) = \pm \frac{nk}{1+e^{kx-n^3k^3 t}}, \tag{21}$$

and the generalized singular solutions

$$u(x,t) = \mp \frac{nk}{1-e^{nkx-n^3k^3 t}}, \tag{22}$$

respectively, where n is finite, and $n \geq 1$.

The obtained kink and singular results are generated for the nonlocal mKdV equation (2), whereas the standard mKdV equation does not possess these kinds of solutions.

5. Singular solutions by using rational hyperbolic functions

To derive another set of solutions, we use the assumption

$$u(x,t) = \frac{\sinh(kx-ct)}{a_0 + a_1\cosh(kx-ct)}. \tag{23}$$

Substituting this assumption in (2), and solving the resulting equation we obtain the following for a_0 and a_1:

$$\begin{aligned} a_0 &= \pm\frac{2}{k}, a_1 = \pm\frac{2}{k}, c = -\frac{1}{2}k^3, \\ a_0 &= \mp\frac{2}{k}, a_1 = \pm\frac{2}{k}, c = -\frac{1}{2}k^3, \end{aligned} \tag{24}$$

which gives the following kink and singular solutions

$$\begin{aligned} u(x,t) &= \pm\frac{k\sinh(kx+\frac{1}{2}k^3t)}{2(1+\cosh(kx+\frac{1}{2}k^3t))}, \\ u(x,t) &= \mp\frac{k\sinh(kx+\frac{1}{2}k^3t)}{2(1-\cosh(kx+\frac{1}{2}k^3t))}, \end{aligned} \tag{25}$$

respectively.

In a like manner, we can assume that the solution takes the form

$$u(x,t) = \frac{\cosh(kx-ct)}{a_0 + a_1\sinh(kx-ct)}. \tag{26}$$

Substituting this assumption in (2), and solving resulting equation we obtain

$$a_0 = 0, a_1 = \pm\frac{1}{k}, c = -2k^3, \tag{27}$$

which gives the following singular solutions

$$u(x,t) = \pm k\coth(kx+2k^3t). \tag{28}$$

The obtained solutions come from this newly integrable nonlocal mKdV equation (2). However, the standard mKdV equation does not give any of the obtained solutions.

6. Periodic solutions

To derive periodic wave solutions, we use the following assumption for $u(x, t)$ in the form

$$u(x,t) = \frac{\sin(kx-ct)}{a_0 + a_1\cos(kx-ct)}. \tag{29}$$

Substituting this assumption in (2), and solving resulting equation we obtain the following set for a_0 and a_1:

$$\begin{aligned} a_0 &= \pm\frac{2}{k}, a_1 = \pm\frac{2}{k}, c = \frac{1}{2}k^3, \\ a_0 &= \mp\frac{2}{k}, a_1 = \pm\frac{2}{k}, c = \frac{1}{2}k^3, \end{aligned} \tag{30}$$

which gives the following periodic and singular solutions

$$\begin{aligned} u(x,t) &= \pm\frac{k\sin(kx-\frac{1}{2}k^3t)}{2(1+\cos(kx-\frac{1}{2}k^3t))}, \\ u(x,t) &= \mp\frac{k\sin(kx-\frac{1}{2}k^3t)}{2(1-\cos(kx-\frac{1}{2}k^3t))}, \end{aligned} \tag{31}$$

respectively.

In a like manner, we can assume that the solution takes the form

$$u(x,t) = \frac{\cos(kx-ct)}{a_0 + a_1\sin(kx-ct)}. \tag{32}$$

Proceeding as before, we obtain the following set for a_0 and a_1:

$$a_0 = 0, a_1 = \pm\frac{1}{k}, c = 2k^3, \tag{33}$$

which gives the following singular solutions

$$u(x,t) = \pm k\cot(kx-2k^3t). \tag{34}$$

7. More periodic solutions

To derive periodic wave solutions, we set

$$u(x,t) = a_0 + a_1\tan(kx-ct), \tag{35}$$

for the solution of nonlocal mKdV equation. Proceeding as before, we obtain

$$a_0 = \beta,\ \beta \text{is any real number}, a_1 = \pm k, c = 6\beta^2k + 2k^3, \tag{36}$$

which gives the following periodic solutions

$$u(x,t) = \beta \pm k\tan(kx-(6\beta^2k+2k^3)t). \tag{37}$$

In a like manner, we can derive the singular solution

$$u(x,t) = \beta \pm k\cot(kx-(6\beta^2k+2k^3)t), \tag{38}$$

8. Discussion

In this work we have performed a detailed study on the integrable nonlocal modified KdV equation. We used a variety of techniques to determine abundant solutions. Moreover, we showed that this equation possesses a family of traveling solitary wave solutions that include, soliton, kink, periodic, and singular solutions. We showed that the nonlocal modified KdV equation, where the nonlocality shows up as sign inversions in both x and t, gives different solutions compared to the classical mKdV equation. The integrable nonlocal modified KdV equation is rich of distinct solutions of different physical structures.

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