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Original Article

The simplified Hirota's method for studying three extended higher-order KdV-type equations

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Abstract

In this work we study three extended higher-order KdV-type equations. The Lax-type equation, the Sawada–Kotera-type equation and the CDG-type equation are derived from the extended KdV equation. We use the simplified Hirota's direct method to derive multiple soliton solutions for each equation. We show that each model gives multiple soliton solutions, where the structures of the obtained solutions differ from the solutions of the canonical form of these equations.

Keywords: Fifth-order KdV equation; Hirota's method; Dispersion relation.

1. Introduction

The celebrated KdV equation is used to model [1–14] the propagation of weakly nonlinear water waves in long, narrow, shallow channels. It also arises in other areas such as hydro magnetic waves in a cold plasma, ion-acoustic waves, and acoustic waves in harmonic crystals. It also incorporates leading-order nonlinearity and dispersion. If the second-order terms are retained then the extended Korteweg-de Vries equation (eKdV) takes the form [1–7]

$$u_t + u_x + \alpha(\lambda u u_x + u_{3x}) + \alpha^2(c_1 u^2 u_x + c_2 u_x u_{2x} + c_3 u u_{3x} + c_4 u_{5x}), \quad (1)$$

where $\alpha \ll 1$ is a non-dimensional measure of the small wave amplitude relative to depth, and the parameters c_1, c_2, c_3, and c_4 are the coefficients of the higher-order order terms, and its values depend on the physical context. The coefficient λ is a non-zero constant. Eq. (1) describes the evolution of steeper waves with shorter wavelengths than in the KdV equation [1,2]. Unlike the standard family of the fifth-order KdV equations, the eKdV includes two linear dispersive terms, namely u_{3x} and u_{5x}, and four nonlinear terms.

It is well known that the KdV equation and the fifth-order KdV equation are completely integrable equations, and both give multiple soliton solutions. The question whether the extended Korteweg-de Vries equation (eKdV) (1) is integrable or not. This question was addressed thoroughly in [1–6]. Marchant et al. [1,2] showed that the extended Eq. (1) becomes a member of the KdV family of integrable equations for the special case

$$c_1 = 1,\ c_2 = \frac{2}{3},\ c_3 = \frac{1}{3},\ c_4 = \frac{1}{3}, \quad (2)$$

and hence it gives multiple soliton solutions. Moreover, Wang et al. [3] examined the integrability of this equation for the specific cases of Lax-type and Sawada–Kotera type forms. In [3], the Bell polynomials approach is used and Bäcklund transformation, and Lax pair for these forms were derived for specific values of the parameters c_i, $1 \le i \le 4$, and the existence of the Lax pair confirms the integrability of (1) for these specific values. In [6], the authors employed the concept of pseudopotential to achieve Lax pair and singularity manifold equation for (1), and its integrability is justified in the sense that it admits the Lax pair. In [4,5], another approach was used to investigate the integrability of (1) for specific values of the parameters.

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In [1,2], Eq. (1) was established and studied for the soliton interaction and the resonant flow of a fluid over typography. In [3], the study for this equation was carried out using Bell polynomials for the derivation of soliton and periodic solutions. However, Dullin et al. [4,5] used the Kodama transformation given in [7,8] to transform the Camassa–Holm equation to the integrable fifth-order KdV equation, referred to, in [4,5], by KdV5, given by

$$u_t + u_x + 3uu_x + 5\alpha^2(uu_{xxx} + 2u_xu_{xx}) + \frac{15}{2}\frac{\lambda^2}{\mu}u^2u_x + \mu(\lambda^2 u_{xxxxx} + u_{xxx}) = 0, \tag{3}$$

which works as a model for the shallow water waves with surface tension, $u(x, t)$ describes the fluid velocity, λ^2 and μ are the length scales [4–5].

Setting $c_1 = 45c_4, c_2 = c_3 = 15c_4, c_4 = \beta, \lambda = 6$ leads to an extended Sawada–Kotera equation (eSK) given as [3]

$$u_t + u_x + \alpha(6uu_x + u_{3x}) + \alpha^2\beta(45u^2u_x + 15u_xu_{2x} + 15uu_{3x} + u_{5x}). \tag{4}$$

However, setting $c_1 = 180c_4, c_2 = c_3 = 30c_4, c_4 = \beta, \lambda = 12$ gives an extended Caudrey–Dodd–Gibbon equation (eCDG) given as

$$u_t + u_x + \alpha(12uu_x + u_{3x}) + \alpha^2\beta(180u^2u_x + 30u_xu_{2x} + 30uu_{3x} + u_{5x}). \tag{5}$$

Moreover, setting $c_1 = 30c_4, c_2 = 20c_4, c_3 = 10c_4, c_4 = \beta, \lambda = 6$ gives an extended Lax equation (eLax) given as

$$u_t + u_x + \alpha(6uu_x + u_{3x}) + \alpha^2\beta(30u^2u_x + 20u_xu_{2x} + 10uu_{3x} + u_{5x}). \tag{6}$$

The extended higher-order KdV Eqs. (4)–(6) involve the two linear dispersive terms u_{3x} and u_{5x} in addition to four nonlinear terms when compared to the standard Sawada–Kotera, Caudrey–Dodd–Gibbon, and Lax equations. Moreover, these equations can reduce to a series of integrable models or can describe such physical phenomena as the amplitude of the shallow-water wave and/or surface wave in fluids [6].

We point out that the extended KdV Eq. (1) describes the evolution of steeper waves with shorter wavelength than in the KdV equation [1,2]. The extended KdV Eq. (1) can reduce to a series of integrable models or can describe such physical phenomena as the amplitude of the shallow-water wave and/or surface wave in fluids [3,6].

The dynamics of shallow water wave flow attracted huge number of works in a variety of fields [15–32]. The reported works were focused on studying a variety of aspects, such as integrability, Lax pairs, Bäcklund transformation, conservation laws, multiple soliton solutions, and various other aspects. Towards these goals, many powerful methods have been used to highlight the various features of the examined equations. Examples of the methods that have been used are the Hirota bilinear method [10], the simplified Hirota's method [11], the Bäcklund transformation method, Darboux transformation, Pfaffian technique, the inverse scattering method, the Painlevé analysis [23,24], the generalized symmetry method, the subsidiary ordinary differential equation method, the coupled amplitude-phase formulation, the sine-cosine method, the sech-tanh method, the mapping and the deformation approach, and many other methods. Hirota's bilinear method [10], and the simplified Hirota's method developed in [11] are the commonly used methods. The simplified Hirota's method [11] does not depend on the construction of the bilinear forms, instead it assumes that the multi-soliton solutions can be assumed as polynomials of exponential functions. The computer symbolic systems such as Maple and Mathematica allow us to perform complicated and tedious calculations.

In this work we plan to use the simplified Hirota's method to formally derive multiple soliton solutions for the extended Sawada–Kotera Eq. (4), the extended CDG Eq. (5), and the extended Lax Eq. (6). Moreover, the work will show the distinct physical structures of the obtained solutions regarding the dispersion relations and the phase shifts as well. We will also show that the extension aspect did not kill the multiple soliton solutions given by the canonical forms.

2. The extended Sawada–Kotera equation

In this section we will study the extended Sawada–Kotera equation (eSK)

$$u_t + u_x + \alpha(6uu_x + u_{3x}) + \alpha^2\beta(45u^2u_x + 15u_xu_{2x} + 15uu_{3x} + u_{5x}). \tag{7}$$

To determine the dispersion relation for (7) we substitute

$$u(x, t) = e^{\theta_i}, \theta_i = k_ix - c_it, \tag{8}$$

into the linear terms of (7) and solve the resulting equation for the dispersion relation c_i to find that

$$c_i = k_i + \alpha k_i^3 + \alpha^2\beta k_i^5, i = 1, 2, 3. \tag{9}$$

Consequently, the phase variables read

$$\theta_i = k_ix - \left(k_i + \alpha k_i^3 + \alpha^2\beta k_i^5\right)t,\ i = 1, 2, 3. \tag{10}$$

To determine the single soliton solution, we use the transformation

$$u(x, t) = R(\ln f(x))_{xx}, \tag{11}$$

where the auxiliary function $f(x, t)$, for the single soliton solution is given by

$$f(x, t) = 1 + e^{k_1x-\left(k_1+\alpha k_1^3+\alpha^2\beta k_1^5\right)t}. \tag{12}$$

Substituting (11) and (12) into (7) and solving for R we find

$$R = 2. \tag{13}$$

This in turn gives the single soliton solution as

$$u(x, t) = \frac{2k_1^2\, e^{k_1x-\left(k_1+\alpha k_1^3+\alpha^2\beta k_1^5\right)t}}{(1 + e^{k_1x-(k_1+\alpha k_1^3+\alpha^2\beta k_1^5)t})^2}. \tag{14}$$

Fig. 1 below shows the soliton solution (14) for $k_1 = 1, \alpha = 1, \beta = 1, -3 \le x \le 3, -3 \le t \le 3$

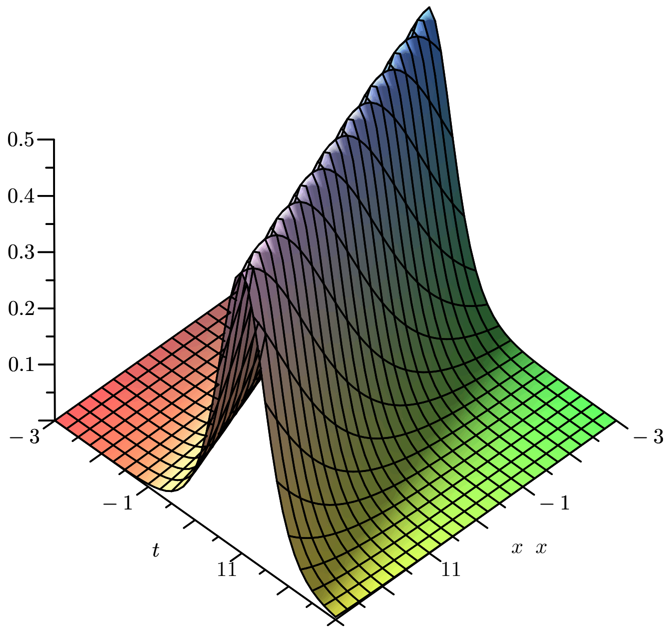

Fig. 1. The soliton solution $u(x, t)$ for $k_1 = 1, \alpha = 1, \beta = 1, -3 \le x \le 3, -3 \le t \le 3$.

For the two soliton solutions we set the auxiliary function as

$$f(x,t) = 1 + e^{\theta_1} + e^{\theta_2} + a_{12}e^{\theta_1+\theta_2}, \tag{15}$$

where the phase variables $\theta_i, i = 1, 2, 3$ are given earlier in (10), and a_{12} is the phase shift that will be determined. Substituting (15) and (11) in (7) and solving for the phase shift a_{12}, we find

$$a_{12} = \frac{(k_1 - k_2)^2\big(5\alpha\beta(k_1^2 - k_1k_2 + k_2^2) + 3\big)}{(k_1 + k_2)^2\big(5\alpha\beta(k_1^2 + k_1k_2 + k_2^2) + 3\big)}, \tag{16}$$

which can be generalized to

$$a_{ij} = \frac{(k_i - k_j)^2(5\alpha\beta(k_i^2 - k_ik_j + k_j^2) + 3)}{(k_i + k_j)^2(5\alpha\beta(k_i^2 - k_ik_j + k_j^2) + 3)}, 1 \le i < j \le 3, \tag{17}$$

provided that

$$(k_i + k_j)^2\big(5\alpha\beta(k_i^2 - k_ik_j + k_j^2) + 3\big) \ne 0. \tag{18}$$

It is obvious that the phase shifts are affected by the parameters α and β as shown. The obtained phase shifts are different than the phase shifts of the standard Sawada–Kotera equation which is usually given by

$$a_{ij} = \frac{(k_i - k_j)^2\left(k_i^2 - k_ik_j + k_j^2\right)}{(k_i + k_j)^2\left(k_i^2 - k_ik_j + k_j^2\right)}, 1 \le i < j \le 3. \tag{19}$$

Substituting (15) and (16) into (11) gives the two soliton solutions for the extended Sawada–Kotera Eq. (7).

It is interesting to point out that Eq. (7) does not show any resonant phenomenon because the phase shift term a_{ij} cannot be 0 or ∞ for $|k_1| \ne |k_2|$.

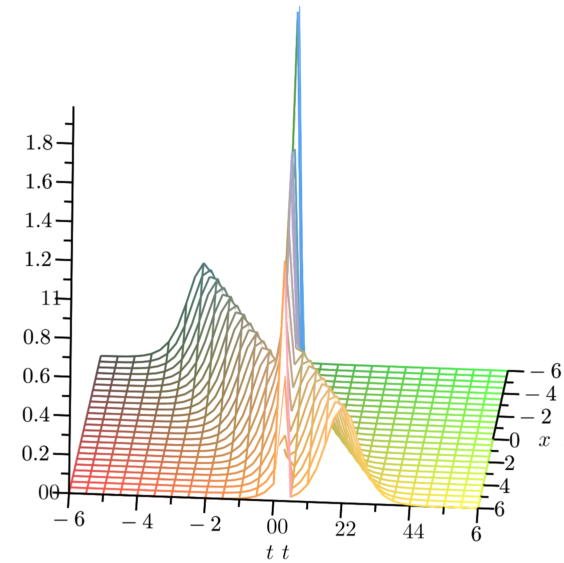

Fig. 2. The two soliton solutions $u(x, t)$ for $k_1 = 1, k_2 = 2, \alpha = 1, \beta = 1, -6 \le x \le 6, -6 \le t \le 6$.

It is worth noting the obtained two-soliton solutions possess distinct physical structures when compared with the two soliton solutions of the standard Sawada–Kotera equation due to the change in phase shifts.

Fig. 2 below shows the two soliton solutions for $k_1 = 1, k_2 = 2, \alpha = 1, \beta = 1, -6 \le x \le 6, -6 \le t \le 6$

For the three soliton solutions, we set the auxiliary function by

$$f(x,t) = 1 + e^{\theta_1} + e^{\theta_2} + e^{\theta_3} + a_{12}e^{\theta_1+\theta_2} + a_{13}e^{\theta_1+\theta_3} + a_{23}e^{\theta_2+\theta_3} + b_{123}e^{\theta_1+\theta_2+\theta_3}. \tag{20}$$

Proceeding as before, we find

$$b_{123} = a_{12}a_{23}a_{13}. \tag{21}$$

The three soliton solutions are obtained by substituting (20) into (11). This shows that the eSK Eq. (7) gives N-soliton solutions for finite N, where $N \ge 1$. The obtained three soliton solutions have distinct physical structures compared to that obtained by the standard SK equation.

Fig. 3 below shows the three soliton solutions for $k_1 = 1, k_2 = 1.2, k_3 = 1.4, \alpha = 1, \beta = 1, -10 \le x \le 10, -10 \le t \le 10$

3. The extended Caudrey–Dodd–Gibbon equation

In this section we will study the extended Caudrey–Dodd–Gibbon equation (eCDG)

$$u_t + u_x + \alpha(12uu_x + u_{3x}) + \alpha^2\beta(180u^2u_x + 30u_xu_{2x} + 30uu_{3x} + u_{5x}). \tag{22}$$

Although the standard Sawada–Kotera equation and the standard Caudrey–Dodd–Gibbon equation were derived on the same basis, but the coefficients of the nonlinear terms in both equations are different.

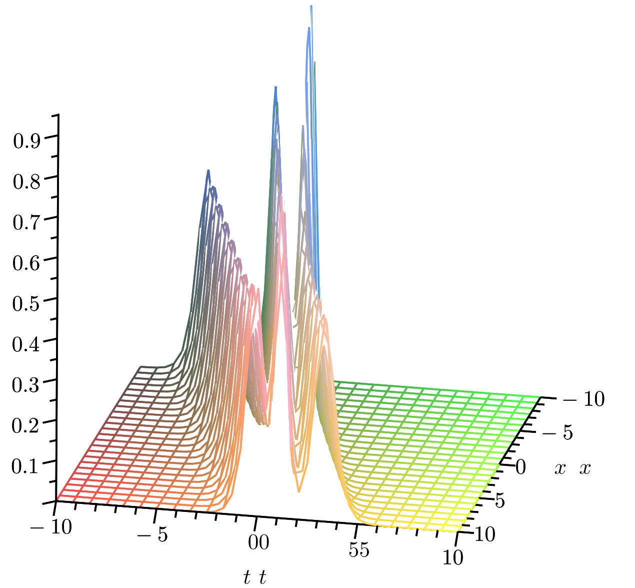

Fig. 3. The three soliton solutions $u(x, t)$ for $k_1 = 1, k_2 = 1.2, k_3 = 1.4, \alpha = 1, \beta = 1, -10 \le x \le 10, -10 \le t \le 10$.

To determine the dispersion relation for (22) we substitute

$$u(x,t) = e^{\theta_i}, \theta_i = k_i x - c_i t, \tag{23}$$

into the linear terms of (22) and proceeding as before to get the dispersion relation c_i as

$$c_i = k_i + \alpha k_i^3 + \alpha^2 \beta k_i^5,\ i = 1, 2, 3. \tag{24}$$

Consequently, the phase variables read

$$\theta_i = k_i x - \left(k_i + \alpha k_i^3 + \alpha^2 \beta k_i^5\right) t, i = 1, 2, 3. \tag{25}$$

To determine the single soliton solution, we use the transformation

$$u(x,t) = R(\ln f(x))_{xx}, \tag{26}$$

where the auxiliary function $f(x, t)$, for the single soliton solution is given by

$$f(x,t) = 1 + e^{k_1 x - \left(k_1 + \alpha k_1^3 + \alpha^2 \beta k_1^5\right) t}. \tag{27}$$

Substituting (26) and (27) into (22) and solving for R we find

$$R = 1. \tag{28}$$

This in turn gives the single soliton solution as

$$u(x,t) = \frac{k_1^2\, e^{k_1 x - \left(k_1 + \alpha k_1^3 + \alpha^2 \beta k_1^5\right) t}}{(1 + e^{k_1 x - (k_1 + \alpha k_1^3 + \alpha^2 \beta k_1^5) t})^2}. \tag{29}$$

For the two soliton solutions we set the auxiliary function as

$$f(x,t) = 1 + e^{\theta_1} + e^{\theta_2} + a_{12} e^{\theta_1 + \theta_2}, \tag{30}$$

where the phase variables $\theta_i, i = 1, 2, 3$ are given earlier in (25), and a_{12} is the phase shift that will be determined. Substituting (30) and (26) in (22) and solving for the phase shift a_{12}, we find

$$a_{12} = \frac{(k_1 - k_2)^2\left(5\alpha\beta(k_1^2 - k_1 k_2 + k_2^2) + 3\right)}{(k_1 + k_2)^2\left(5\alpha\beta(k_1^2 + k_1 k_2 + k_2^2) + 3\right)}, \tag{31}$$

which can be generalized to

$$a_{ij} = \frac{(k_i - k_j)^2 (5\alpha\beta(k_i^2 - k_i k_j + k_j^2) + 3)}{(k_i + k_j)^2 (5\alpha\beta(k_i^2 - k_i k_j + k_j^2) + 3)}, 1 \le i < j \le 3. \tag{32}$$

It is obvious that the phase shifts are affected by the parameters α and β as shown. The obtained phase shifts are different than the phase shifts of the standard Caudrey–Dodd–Gibbon equation which is usually given by

$$a_{ij} = \frac{(k_i - k_j)^2 (k_i^2 - k_i k_j + k_j^2)}{(k_i + k_j)^2 (k_i^2 - k_i k_j + k_j^2)}, 1 \le i < j \le 3. \tag{33}$$

It is interesting to point out that Eq. (7) does not show any resonant phenomenon because the phase shift term a_{ij} cannot be 0 or ∞ for $|k_1| \neq |k_2|$.

Substituting (30) and (31) into (26) gives the two soliton solutions for the extended Caudrey–Dodd–Gibbon Eq. (22). It is worth noting the obtained two-soliton solutions possess distinct physical structures when compared with the two soliton solutions of the standard Caudrey–Dodd–Gibbon equation due to the change in phase shifts.

For the three soliton solutions, we set the auxiliary function by

$$\begin{aligned} f(x,t) = 1 + e^{\theta_1} + e^{\theta_2} + e^{\theta_3} + a_{12} e^{\theta_1 + \theta_2} + a_{13} e^{\theta_1 + \theta_3} \\ + a_{23} e^{\theta_2 + \theta_3} + b_{123} e^{\theta_1 + \theta_2 + \theta_3}. \end{aligned} \tag{34}$$

Proceeding as before, we find

$$b_{123} = a_{12} a_{23} a_{13}. \tag{35}$$

The three soliton solutions are obtained by substituting (34) into (26). This shows that the eCDG Eq. (22) gives N-soliton solutions for finite N, where $N \ge 1$. The obtained three solitons solutions have distinct physical structures compared to that obtained by the standard SK equation.

It is worth noting that the solutions of the eCDG equation are different than the solutions obtained for the eSK equation because of the difference in the coefficient R for both equations. The amplitude of the solutions for the eSK equation is twice than that of the eCDG equation.

4. The extended Lax equation

In this section we will study the extended Lax equation (eLax)

$$\begin{aligned} u_t + u_x + \alpha(6uu_x + u_{3x}) \\ + \alpha^2 \beta (30u^2 u_x + 20u_x u_{2x} + 10uu_{3x} + u_{5x}). \end{aligned} \tag{36}$$

To determine the dispersion relation for (36) we substitute

$$u(x,t) = e^{\theta_i}, \theta_i = k_i x - c_i t, \tag{37}$$

into the linear terms of (36) we find the dispersion relation c_i as

$$c_i = k_i + \alpha k_i^3 + \alpha^2 \beta k_i^5, i = 1, 2, 3. \tag{38}$$

Consequently, the phase variables read

$$\theta_i = k_i x - \left(k_i + \alpha k_i^3 + \alpha^2 \beta k_i^5\right) t, i = 1, 2, 3. \tag{39}$$

To determine the single soliton solution, we use the transformation

$$u(x,t) = R(\ln f(x))_{xx}, \tag{40}$$

where the auxiliary function $f(x, t)$, for the single soliton solution is given by

$$f(x,t) = 1 + e^{k_1 x - \left(k_1 + \alpha k_1^3 + \alpha^2 \beta k_1^5\right) t}. \tag{41}$$

Substituting (40) and (41) into (36) and solving for R we find

$$R = 2. \tag{42}$$

This in turn gives the single soliton solution as

$$u(x,t) = \frac{2k_1^2\, e^{k_1 x - \left(k_1 + \alpha k_1^3 + \alpha^2 \beta k_1^5\right) t}}{(1 + e^{k_1 x - (k_1 + \alpha k_1^3 + \alpha^2 \beta k_1^5) t})^2}. \tag{43}$$

For the two soliton solutions we set the auxiliary function as

$$f(x,t) = 1 + e^{\theta_1} + e^{\theta_2} + a_{12} e^{\theta_1 + \theta_2}, \tag{44}$$

where the phase variables $\theta_i, i = 1, 2, 3$ are given earlier in (39), and a_{12} is the phase shift that will be determined. Substituting (44) in (36) and solving for the phase shift a_{12}, we find

$$a_{12} = \frac{(k_1 - k_2)^2}{(k_1 + k_2)^2}, \tag{45}$$

which can be generalized to

$$a_{ij} = \frac{(k_i - k_j)^2}{(k_i + k_j)^2}, 1 \leq i < j \leq 3. \tag{46}$$

Unlike the eSK and the eCDG equations, the phase shifts are not affected by the parameters α and β. The obtained phase shifts are exactly the same phase shifts of the standard Lax equation.

It is interesting to point out that Eq. (7) does not show any resonant phenomenon because the phase shift term a_{ij} cannot be 0 or ∞ for $|k_1| \neq |k_2|$.

Substituting (44) and (45) into (40) gives the two soliton solutions for the extended Lax Eq. (36). It is worth noting the obtained two-soliton solutions possess distinct physical structures when compared with the two soliton solutions of the standard Lax equation due to the existence of the parameters α and β.

For the three soliton solutions, we set the auxiliary function by

$$f(x,t) = 1 + e^{\theta_1} + e^{\theta_2} + e^{\theta_3} + a_{12} e^{\theta_1 + \theta_2} + a_{13} e^{\theta_1 + \theta_3} + a_{23} e^{\theta_2 + \theta_3} + b_{123} e^{\theta_1 + \theta_2 + \theta_3}. \tag{47}$$

Proceeding as before, we find

$$b_{123} = a_{12} a_{23} a_{13}. \tag{48}$$

The three soliton solutions are obtained as presented earlier. This shows that the eLax Eq. (7) gives N-soliton solutions for finite N, where $N \geq 1$.

5. Discussion

In this work we studied three extended higher-order KdV-type equation, namely the eSK equation, the eCDG equation, and the eLax equation. We showed that the three extended equations give multiple soliton solutions with distinct physical structures. Although the dispersion relations are the same for the three extended equations, but the soliton solutions are distinct. Moreover, the eSK and the eCDG equations possess the same phase shift, which is affected by the parameters α and β, whereas the eLax equation retained the same phase shift as the standard Lax equation. We used the simplified Hirota's method to formally derive the multiple soliton solutions for the three extended equation.

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