Available online at www.sciencedirect.com

ScienceDirect

Journal of Ocean Engineering and Science 1 (2016) 135–148

www.elsevier.com/locate/joes

Wave scattering by a horizontal circular cylinder in a three-layer fluid

Dilip Das*

Department of Mathematics, Shibpur Dinobundhoo Institution (College), Calcutta University, Howrah, West Bengal, India

Received 5 August 2015; received in revised form 19 October 2015; accepted 18 November 2015
Available online 2 April 2016

Abstract

Using linear water-wave theory, wave scattering by a horizontal circular cylinder submerged in a three-layer ocean consisting of a layer of finite depth bounded above by finite depth water with free surface and below by an infinite layer of fluid of greater density is considered. The cylinder is submerged in either of the three layers. In such a situation time-harmonic waves with given frequency can propagate with three different wave numbers. Employing the method of multipoles the problem is reduced to an infinite system of linear equations which are solved numerically by standard technique after truncation. The transmission and reflection coefficients are obtained and depicted graphically against the wave number for all cases. In a two-layer fluid there are energy identities that exist connecting the transmission and reflection coefficients that arise. These energy identities are systematically extended to the three-fluid cases which are obtained.

Keywords: Three-layer fluid; Wave scattering; Circular cylinder; Reflection and transmission coefficients; Energy identities.

1. Introduction

The propagation of waves in layered fluids is a well-documented phenomenon such a situation may arise when the continuous density profile of an ocean or atmosphere has been approximated to one made up of multiple horizontal layers of constant density. Typically system of two or three layers is considered in literature of water wave theory. Stokes [1] first investigated wave propagation in a two-layer fluid with a free surface and an interface. For such a two-layer fluid it is known that time-harmonic waves with a given frequency can propagate with two different wave numbers [2]. Linton and McIver [3] developed a general theory for two dimensional wave propagation in such a two-layer fluid. Due to the presence of an obstacle, an incident wave of a particular wave number gets reflected and transmitted into waves of both the wave numbers, so that on scattering by an obstacle, transfer of energy from one mode to another take place. The problem arose in connection with modeling an underwater pipe bridge across the Norwegian fjords consisting of a layer of fresh water on the top of a deep layer of salt water. Cadby and Linton [4] studied three-dimensional wave scattering and radiation by a submerged sphere in a two-layer fluid by using the method of multipoles and Linton and Cadby [5] also investigated the problem of scattering of obliquely incident waves by a long circular cylinder in a two-layer fluid. Various aspect of wave motion has been analyzed in the two-layer fluids [6–9].

A more general class of problems in two-layer fluid medium is the wave interaction with floating elastic plate or floating ice-sheet, which are of ocean engineering interest. The use of floating elastic plate in the hydroelastic analysis of very large floating structure and the wave interaction with floating ice-sheet is well known in the literature. Das and Mandal [10,11] studied the water wave scattering by a horizontal circular cylinder in two-layer fluid with an ice-cover. Das [12] considered the solution of the dispersion equation for internal waves in two-layer fluid with an ice-cover and also Das and Mandal [13] investigated the water wave radiation by a sphere in two-layer fluid with an ice-cover. Recently Das and Thakur [14] studied the wave scattering by a sphere in the two-layer fluid with an ice-cover.

* Tel.: +91 9433567457 (mobile).
E-mail address: dilipdas99@gmail.com.

http://dx.doi.org/10.1016/j.joes.2016.03.001

More recently, however, interest has been extended to bodies which are immersed in three-layer fluids, each fluid having a different density. Some interesting results also have been published for the case of three-layer fluids. This is a particular interest in understanding wave transformation in the presence of floating or submerged structures in continental shelves and estuaries. Such sharp density gradients can be generated in the ocean due to gravitational settling of sediments carried by a fluids or by solar heating of the upper layer or in an estuary or a fjord in to which fresh river water flows over oceanic water, which is more saline and consequently heavier. Michallet and Dias [15] have considered waves in three-layer systems that contain rigid horizontal walls above the upper most fluid and below the lower most one. The linear stability for three-layer fluid has been analyzed by Taylar [16]. Chakrabarti et al. [17] studied the trapped modes of waves in three-layer fluid in a channel when a cylinder is totally submerged in the lower layer fluid. Chen and Forbes [18] investigated the steady periodic waves in a three-layer fluid with shear in the middle layer. Recently Mondal and Sahoo [19] studied wave structure interaction problems in three-layer fluids. Less work has been done on the study of wave structure interaction problems in the three-layer fluid in which the upper surface of the upper fluid is free and having two interfaces. In this situation, there are three possible linear wave systems at a given frequency, each with different wave number, one mode corresponds to an oscillation which is mainly confined to the upper fluid, second mode may be thought of as an oscillation in the middle fluid and the third mode corresponds to an oscillation which is confined to the lower layer fluid. For an arbitrary but stable density ratio, there is the possibility that some of the energy may be transferred from one mode to another if the wave field interacts with a body. Thus the three-layer model may be considered as a more accurate realization of the two-layer model, which is being considered in various models.

The wave scattering for an arbitrary, two-dimensional configuration of horizontal circular cylinder in a three-layer fluid are considered here. Under the linear theory of water waves, the various hydrodynamic relations which connect various scattering and radiation quantities have been derived by many authors over the years. These relations may be obtained by applying Greens theorem to two different potentials and a systematic derivation of all the first-order relations is given by Newman [20] in a single-layer fluid. Also all the relations for a two-layer fluid are systematic derived from Greens theorem [3–5,11,21]. Here in Section 2, following the approach of Newman [20], the relations energy identities for a three-layer fluid are derived from Green's theorem which is used as a check on the corrections of all numerical results for the reflection and transmission coefficients. In Section 2.1, the wave scattering by a single, horizontal circular cylinder which is totally contained in the lower layer fluid is investigated. It is well known [3] that there is zero reflection of waves of any frequency by a circular cylinder submerged in a lower layer of a two-layer fluid and it is interesting to investigate the corresponding situation in the three-layer fluid. This analysis is slightly more complicated for the three-layer fluid because there are three possible reflected waves of different wave numbers to consider. However, by using the multipoles expansion, it is found that regardless of the wave number of the incident wave, there is no reflection of energy into either mode. The transmission coefficients are investigated numerically which the aid of the energy identity relations derived in Section 2. The analysis is repeated in Section 2.2 for a circular cylinder which is considered in the middle layer and Section 2.3 for a circular cylinder in the upper layer. Reflections of wave are found to occur in both the cases. Numerical estimates for the reflection and transmission coefficients are obtained and are depicted graphically against the wave number in a number of figures in both the cases and also the reflection and transmission coefficients are investigated numerically which the aid of the energy identity relations derived in Section 2.

2. Scattering problem in a three-layer

It is here concerned with irrotational motion in three superposed non-viscous incompressible fluids under the action of gravity, neglecting any effect due to surface tension at the interfaces of the three fluids, the upper being of finite depth H and the middle layer being of finite depth h, while the lower layer being infinitely deep. The upper, middle and lower layer fluids have densities ρ_1, ρ_2 and $\rho_3(\rho_1 < \rho_2 < \rho_3)$, respectively. Cartesian co-ordinates are chosen such that (x, z)-plane coincides with the undisturbed interface between the middle and lower layer (ML) fluids. The y-axis points vertically upwards with $y = 0$ as the mean position of the interface of the ML fluid, $y = h(> 0)$ as the mean position of the interface of the upper and middle (UM) fluid and $y = H + h(> 0)$ as the mean position of the linearized free surface. Under the usual assumptions of linear water wave theory a velocity potential can be defined for waves in the form $\mathrm{Re}\{\phi(x, y)e^{-i\sigma t}\}$ where $\phi(x, y)$ is a complex valued potential function, σ is the angular frequency.

The upper fluid, $h < y < H + h$, will be referred to as region I,the middle fluid, $0 < y < h$, will be referred to as region II, while the lower fluid, $y < 0$, will be referred to as region III. The potential in the upper fluid will be denoted by ϕ^I and that in the middle and lower fluids by ϕ^{II}, ϕ^{III} respectively. ϕ^I, ϕ^{II} and ϕ^{III} satisfied Laplace's equation

$$\nabla^2\phi^i = 0, \quad i = I, II, III. \tag{1}$$

Linearized boundary conditions on the interfaces and at the free surface are

$$\phi_y{}^I = \phi_y{}^{II} \quad \text{on} \quad y = h, \tag{2}$$

$$s_1(\phi_y{}^I - K\phi^I) = \phi_y{}^{II} - K\phi^{II} \quad \text{on} \quad y = h, \tag{3}$$

$$\phi_y{}^{II} = \phi_y{}^{III} \quad \text{on} \quad y = 0, \tag{4}$$

$$s_2(\phi_y{}^{II} - K\phi^{II}) = \phi_y{}^{III} - K\phi^{III} \quad \text{on} \quad y = 0, \tag{5}$$

where

$$s_1 = \frac{\rho_1}{\rho_2}(< 1) \quad \text{and} \quad s_2 = \frac{\rho 2}{\rho_3}(< 1),$$

$$\phi_y{}^I - K\phi^I = 0, \text{ on } y = H + h, \tag{6}$$

where $K = \sigma^2/g$. The boundary conditions (2) and (4) are obtained from the continuity of normal velocity at the interface between UM and ML fluids respectively, while conditions (3) and (5) are obtained from the continuity of pressure at the interface between UM and ML fluids respectively. Also condition at large depth is

$$\nabla\phi^{III} \to 0 \quad \text{as} \quad y \to -\infty. \tag{7}$$

In three-layer fluid progressive waves have the form (except for a multiplicative constant)

$$\phi^I = e^{\pm ikx}\{(K+k)e^{k(y-h-H)} + (K-k)e^{-k(y-h-H)}\}, \tag{8}$$

$$\phi^{II} = e^{\pm ikx}\{M_1 e^{k(y-h)} + M_2 e^{-k(y-h)}\}, \tag{9}$$

$$\phi^{III} = e^{\pm ikx+ky}\{M_1 e^{kh} - M_2 e^{-kh}\}, \tag{10}$$

where

$$M_{1,2} = \frac{k \pm K}{2K}[\pm\{(K \pm k) - s_1(k \mp K)\}e^{\mp kH} \mp (k \mp K)(1-s_1)e^{\pm kH}],$$

where k satisfies the dispersion equation

$$H(k) \equiv (k-K)[(k+K)\{(k+K\sigma_1)e^{-2kH} - (k-K)\}e^{-2kh} - (k-K\sigma_2)\{(k+K)e^{-2kH} - (k-K\sigma_1)\}] = 0, \tag{11}$$

where

$$\sigma_1 = \frac{1+s_1}{1-s_1} \quad \text{and} \quad \sigma_2 = \frac{1+s_2}{1-s_2}.$$

It follows that the dispersion Eq. (11) has exactly three positive real roots K, k_1 and k_2 ($k_2 > k_1$) (say), k_1, k_2 satisfy the equation

$$h(k) = (k+K)\{(k+K\sigma_1)e^{-2kH} - (k-K)\}e^{-2kh} - (k-K\sigma_2)\{(k+K)e^{-2kH} - (k-K\sigma_1)\} = 0. \tag{12}$$

Thus for the case $k = K$, progressive waves are of the form

$$\phi^I = \phi^{II} = \phi^{III} = 2Ke^{\pm iKx + K(y-h-H)}. \tag{13}$$

Also for the case $k = k_j$, $(j = 1, 2)$ progressive waves are thus

$$\phi^I = e^{\pm ik_j x}\{(K+k_j)e^{k_j(y-h-H)} + (K-k_j)e^{-k_j(y-h-H)}\}, \tag{14}$$

$$\phi^{II} = e^{\pm ik_j x}\{M_1^j e^{k_j(y-h)} + M_2^j e^{-k_j(y-h)}\}, \tag{15}$$

$$\phi^{III} = e^{\pm ik_j x + k_j y}\{M^j{}_1 e^{k_j h} - M^j{}_2 e^{-k_j h}\}, \tag{16}$$

where

$$M^j{}_{1,2} = \frac{k_j \pm K}{2K}\big[\pm\{(K \pm k_j) - s_1(k_j \mp K)\}e^{\mp k_j H} \mp (k_j \mp K)(1-s_1)e^{\pm k_j H}\big].$$

Waves of all wave numbers can exist and they can propagate in either direction. In any wave scattering problem therefore, the far-field will take the form of incoming and outgoing waves at each of the wave numbers $K, k_j (j = 1, 2)$. It is given by

$$\begin{aligned}\phi^{I,II,II} \sim\ & A^{\pm}e^{\pm iKx+Ky} + B^{\pm}e^{\pm ik_1x}(g_1^1(y), g_2^1(y), e^{k_1y}) \\ & + C^{\pm}e^{\pm ik_2x}(g_1^2(y), g_2^2(y), e^{k_2y}) + D^{\pm}e^{\mp iKx+Ky} \\ & + E^{\pm}e^{\mp ik_1x}(g_1^1(y), g_2^1(y), e^{k_1y}) \\ & + F^{\pm}e^{\mp ik_2x}(g_1^2(y), g_2^2(y), e^{k_2y}),\end{aligned} \tag{17}$$

as $x \to \pm\infty$, where

$$g_1^j(y) = 2\frac{(k_j+K)e^{-k_j(2h+2H-y)} + (k_j-K)e^{-k_jy}}{(1-s_1)(1-\sigma_2)\{(k_j+K)e^{-2k_jH} - (k_j - K\sigma_1)\}},$$

$j = 1, 2,$

$$g_2^j(y) = 2\frac{(k_j - K\sigma_2)e^{k_jy} + (k_j-K)e^{-k_jy}}{K(1-\sigma_2)}, \quad j = 1, 2.$$

Convenient shorthand for (17) is

$$\phi \sim \{A^-, B^-, C^-, D^-, E^-, F^-; A^+, B^+, C^+, D^+, E^+, F^+\}. \tag{18}$$

In the case of a single-layer fluid, for any scattering problem, the reflection and transmission coefficients satisfy the energy identity, which is generally used as a partial check on the correctness of the analytical or computed values of these coefficients. For a two-layer fluid with a free surface, there exists two energy identities corresponding to scattering of incident waves of two different wave numbers [3]. These energy identities were derived by appropriate uses of Green's integral theorem. For a three-layer fluid with a free surface, energy identities are derived here by using the appropriate Green's integral theorem. These identities are used here as partial numerical checks for all the data points in obtaining the various curves for the reflection and transmission coefficients.

There are a number of bodies in a three-layer fluid is considered here, some in the upper layer, some in the middle layer, some in the lower layer and some standing the UM and ML. The boundaries of these bodies lying in the upper fluid will be denoted by B_I, and those in the middle and lower fluids by B_{II} and B_{III} respectively. Assume that ϕ and ψ are solutions to two different problems, with $\frac{\partial\phi}{\partial n}$ and $\frac{\partial\psi}{\partial n}$ given in the boundaries B_I, B_{II} and B_{III}, with the far field form of ϕ given by (18) and

$$\phi \sim \{M^-, N^-, P^-, Q^-, R^-, S^-; M^+, N^+, P^+, Q^+, R^+, S^+\}. \tag{19}$$

To obtain the energy identities, apply Green's integral theorem, which for harmonic functions ϕ and ψ takes the form

$$\int_S \left(\phi\frac{\partial\psi}{\partial n} - \psi\frac{\partial\phi}{\partial n}\right)ds = 0, \tag{20}$$

where S denotes the boundary of the fluid region completely contained in one of the fluid layers and $\partial/\partial n$ is the derivative with respect to the outward normal.

First we choose S in (20) to be the boundary of the region in the upper fluid bounded internally by $x = \pm X, h \le y \le h + H$; $y = h + H, |x| \le X$; $y = h, |x| \le X$ and externally by the body boundary B_I, and ultimately make $X \to \infty$, and next to be the boundary of the region in the middle fluid bounded internally by $x=\pm X, 0\le y\le h; y=0, |x|\le X; y=h, |x|\le X$ and externally by the body boundary B_{II}, and ultimately make $X\to\infty$, and finally to be the boundary of the region in the lower fluid bounded internally by $x=\pm X, -Y\le y\le 0; y=-Y, |x|\le X; y=0, |x|\le X$ and externally by the body boundary B_{III}, and ultimately make both $X, Y\to\infty$.

After using the results

$$s_{1,2}\left(\phi^{I,II}\frac{\partial}{\partial y}\psi^{I,II}-\psi^{I,II}\frac{\partial}{\partial y}\phi^{I,II}\right)=\phi^{II,III}\frac{\partial}{\partial y}\psi^{II,III}-\psi^{II,III}\frac{\partial}{\partial y}\phi^{II,III}$$

at the interfaces between UM and ML respectively and also use the results

$$s_1s_2\int_h^{h+H}g_1^{1,2}(y)e^{Ky}dy+s_2\int_0^h g_2^{1,2}(y)e^{Ky}dy+\int_{-\infty}^0 e^{(K+k_{1,2})y}dy=0,$$

$$s_1s_2\int_h^{h+H}g_1^1(y)g_1^2(y)dy+s_2\int_0^h g_2^1(y)g_2^2(y)dy+\int_{-\infty}^0 e^(k_1+k_2)ydy=0,$$

then obtain after some lengthy algebra

$$\begin{aligned}&\int_{B_I}s_1s_2\left(\phi^I\frac{\partial\psi^I}{\partial n}-\psi^I\frac{\partial\phi^I}{\partial n}\right)ds+\int_{B_{II}}s_2\left(\phi^{II}\frac{\partial\psi^{II}}{\partial n}-\psi^{II}\frac{\partial\phi^{II}}{\partial n}\right)ds+\int_{B_{III}}\left(\phi^{III}\frac{\partial\psi^{III}}{\partial n}-\psi^{III}\frac{\partial\phi^{III}}{\partial n}\right)ds\\&=J_K(A^+Q^+-D^+M^++A^-Q^--D^-M^-)\\&+J_{k_1}(B^+R^+-E^+N^++B^-R^--E^-N^-)\\&+J_{k_2}(C^+S^+-F^+P^++C^-S^--F^-P^-),\end{aligned}\tag{21}$$

where

$$J_K=i\left[1+2Ks_2\left(s_1\int_h^{h+H}e^{2Ky}dy+\int_0^h e^{2Ky}dy\right)\right],$$

$$J_{k_j}=i\left[1+2k_js_2\left(s_1\int_h^{h+H}(g_j^1(y))^2dy+\int_0^h(g_j^1(y))^2dy\right)\right],\quad j=1,2.$$

Since ϕ and ψ are both the scattering potentials having zero normal derivatives on all body boundaries then the left-hand side of (21) is zero. If we consider the scattering of waves by a fixed sets of bodies, then there are six problems to consider. These are the scattering of an incident wave of wavenumber K from $x=-\infty$, which will refer to as problem 1; the scattering of an incident wave of wavenumber K from $x=\infty$ (problem 2); the scattering of an incident wave of wavenumber k_1 from $x=-\infty$ (problem 3); the scattering of an incident wave of wavenumber k_1 from $x=\infty$ (problem 4); the scattering of an incident wave of wavenumber k_2 from $x=-\infty$ (problem 5); and the scattering of an incident wave of wavenumber k_2 from $x=\infty$ (problem 6). In each case there may be reflected and transmitted waves of wave numbers K, k_1, k_2. R_K, R_{k_1}, R_{k_2} will be represent reflection coefficients corresponding to the wave of wavenumbers K, k_1, k_2 respectively and similarly T_K, T_{k_1}, T_{k_2} for the transmission coefficients corresponding to the wave of wavenumbers K, k_1, k_2, respectively. Thus the six problems are characterized using the notation of (18) by

$$\phi_j\sim\left\{R_K^j, R_{k_1}^j, R_{k_2}^j, \delta_1^j, \delta_3^j, \delta_5^j; T_K^j, T_{k_1}^j, T_{k_2}^j, \delta_2^j, \delta_4^j, \delta_6^j\right\},\quad j=1,2,3,4,5,6,$$

where δ_i^j is delta function. Applying (20) to ϕ_j and its complex conjugate $\overline{\phi_j}$ lead to

$$\begin{aligned}&|R_K^j|^2+|T_K^j|^2+J_1\left(|R_{k_1}^j|^2+|T_{k1}^j|^2\right)+J_2\left(|R_{k_2}^j|^2+|T_{k_2}^j|^2\right)\\&=\delta_1^j+\delta_2^j+J_1\left(\delta_3^j+\delta_4^j\right)+J_2\left(\delta_5^j+\delta_6^j\right),\\&j=1,2,3,4,5,6,\end{aligned}\tag{22}$$

where $J_1=J_{k_1}/J_K$ and $J_2=J_{k_2}/J_K$. Relations (22) are called the *energy identities.*

Taking all possible pairs of the functions from ϕ_j, $(j=1,2,3,4,5,6)$ and applying (18) leads to the relations

$$T_K^1=T_K^2;\quad T_{k_{1,2}}^{3,5}=T_{k_{1,2}}^{4,6};\quad T_K^{3,4}=J_1T_{k_1}^{2,1};$$
$$T_K^{5,6}=J_2T_{k2}^{2,1};\quad R_K^{3,4}=J_1R_{k_1}^{1,2};\quad R_K^{5,6}=J_2R_{k_2}^{1,2};$$
$$J_1R_{k_1}^{5,6}=J_2R_{k_2}^{3,4};\quad J_1T_{k_1}^{5,6}=J_2T_{k_2}^{4,3}.$$

Here it is observed that the transmission coefficient for the wave of the same wavenumber as the incident wave is independent to the direction of incidence. This has a direct analogue in the non-stratified fluid case where it is well-known that the transmission coefficient is independent of the direction of the incident wave.

Let $E_{R_K}^1$ be the reflected energy at wavenumber K due to an incident wave of unit energy and wavenumber K from $x=-\infty$ and so on. Then it is convenient to define energies as follows:

$$\begin{aligned}&E_{R_K,T_K}^j=|R_K^j,T_K^j|^2,\quad E_{R_{k_1},T_{k_1}}^j=J_1|R_{k_1}^j,T_{k_1}^j|^2,\\&E_{R_{k_2},T_{k_2}}^j=J_2|R_{k_2}^j,T_{k_2}^j|^2,\quad j=1,2,\\&E_{R_K,T_K}^j=(1/J_1)|R_K^j,T_K^j|^2,\quad E_{R_{k_1},T_{k_1}}^j=|R_{k_1}^j,T_{k_1}^j|^2,\\&E_{R_{k_2},T_{k_2}}^j=(J_2/J_1)|R_{k_2}^j,T_{k_2}^j|^2,\quad j=3,4,\\&E_{R_K,T_K}^j=(1/J_2)|R_K^j,T_K^j|^2, E_{R_{k_1},T_{k_1}}^j=(J_1/J_2)|R_{k_1}^j,T_{k_1}^j|^2,\\&E_{R_{k_2},T_{k_2}}^j=|R_{k_2}^j,T_{k_2}^j|^2,\quad j=5,6.\end{aligned}$$

The energy relations (22) then become

$$\begin{aligned}&E_{R_K}^j+E_{T_K}^j+E_{R_{k_1}}^j+E_{T_{k_1}}^j+E_{R_{k_2}}^j\\&+E_{T_{k_2}}^j=1,\quad j=1,2,3,4,5,6.\end{aligned}\tag{23}$$

In terms of these energies, above equations imply

$$E_{R_K}^{3,5}=E_{R_{k_{1,2}}}^1,\quad E_{R_K}^{4,6}=E_{R_{k_{1,2}}}^2,\quad E_{T_K}^{4,6}=E_{T_{k_{1,2}}}^1,$$

$$E^{3,5}_{T_K} = E^{2}_{T_{k_{1,2}}}, \quad E^{5,6}_{R_{k_1}} = E^{3,4}_{R_{k_2}}, \quad E^{6,5}_{T_{k_1}} = E^{3.4}_{T_{k_2}}.$$

First and second relations can be stated as: the energy reflected at wavenumber K due to an incident wave from $x = -\infty(+\infty)$ of wavenumbers k_1 or k_2 are the same as the energy reflected at wavenumbers k_1 or k_2 due to an incident wave from $x = -\infty(+\infty)$ of wave number K. Third and fourth relations implies: the energy transmitted at wavenumber K due to an incident wave from $x = -\infty(+\infty)$ of wavenumbers k_1 or k_2 are the same as the energy transmitted at wavenumbers k_1 or k_2 due to an incident wave from $x = -\infty(+\infty)$ of wave number K. Finally last two relations implies: the energy reflected at wavenumber k_1 due to an incident wave from $x = -\infty(+\infty)$ of wavenumber k_2 is the same as the energy reflected at wavenumber k_2 due to an incident wave from $x = -\infty(+\infty)$ of wave number k_1 and the energy transmitted at wavenumber k_1 due to an incident wave from $x = -\infty(+\infty)$ of wavenumber k_2 is the same as the energy transmitted at wavenumber k_2 due to an incident wave from $x = -\infty(+\infty)$ of wave number k_1. For the case of a body symmetric about $x = 0$ the direction of the incident wave is immaterial, thus problems 2, 4 and 6 being equivalent to the problems 1, 3 and 5, respectively. Thus the problems 1, 3 and 5 are considered here.

In problem 1 the form of the incident wave of wavenumber K is

$$\phi_{inc} = e^{iKx+Ky}, \tag{24}$$

in problem 3, incident plane wave ϕ_{inc} of wave number k_1 has the form

$$\phi^{I,II,III}_{inc} = e^{ik_1x}\left(g^1_1(y), g^1_2(y), e^{k_1y}\right) \tag{25}$$

and in problem 5 the form of the incident wave of wavenumber k_2 are

$$\phi^{I,II,III}_{inc} = e^{ik_2x}\left(g^2_1(y), g^2_2(y), e^{k_2y}\right). \tag{26}$$

2.1. Cylinder in the lower layer

Let a horizontal circular cylinder of radius a have its axis at $y = f(< 0)$ and its generator runs parallel to z-axis. Polar co-ordinates (r, θ) are defined in the (x, y)-plane by

$$x = r\sin\theta \quad \text{and} \quad y = f - r\cos\theta. \tag{27}$$

It is convenient to distinguish those multipoles symmetric about $x = 0$ and those antisymmetric about this line. These will be denoted by ϕ^s_n and ϕ^a_n respectively. The form of those functions will be different in regions I, II, III.

Solutions of Laplace's equation singular at $y = f < 0$ are $r^{-n}\cos n\theta$and $r^{-n}\sin n\theta, n \geq 1$, and these have the integral representations [22]

$$\frac{\cos n\theta, \sin n\theta}{r^n} = \frac{(-1)^{n,n+1}}{(n-1)!}\int_0^\infty k^{n-1}e^{-k(y-f)}\cos kx, \sin kx\, dk.$$

It is straightforward to add suitable solutions of Laplace's equation to the symmetric and anti-symmetric multipoles so that the boundary conditions (2)–(7) are satisfied. We obtain

$$\phi^{Is,a}_n = \frac{(-1)^{n,n+1}}{(n-1)!}\int_0^\infty k^{n-1}\big(A(k)e^{ky} + B(k)e^{-ky}\big)\cos kx, \sin kx\, dk, \tag{28}$$

$$\phi^{IIs,a}_n = \frac{(-1)^{n,n+1}}{(n-1)!}\int_0^\infty k^{n-1}\big(C(k)e^{ky} + D(k)e^{-ky}\big)\cos kx, \sin kx\, dk, \tag{29}$$

$$\phi^{IIIs,a}_n = \frac{\cos n\theta, \sin n\theta}{r^n} + \frac{(-1)^{n,n+1}}{(n-1)!}\int_0^\infty k^{n-1}E(k)e^{ky}\cos kx, \sin kx\, dk, \tag{30}$$

where

$$A(k) = 4\frac{(k+K)K^2e^{kf}e^{-2k(h+H)}}{(1-s_1)(1-s_2)H(k)}, B(k) = 4\frac{K^2e^{kf}}{(1-s_1)(1-s_2)h(k)},$$

$$C(k) = 2\frac{K(k+K)e^{kf}\left\{(k+K\sigma_1)e^{-2k(h+H)} - (k-K)e^{-2kh}\right\}}{(1-s_2)H(k)},$$

$$D(k) = 2\frac{Ke^{kf}\left\{(k+K)e^{-2kH} - (k-K\sigma_1)\right\}}{(1-s_2)h(k)},$$

$$E(k) = e^{kf}[(k+K\sigma_2)\left\{(k+K\sigma_1)e^{-2k(h+H)} - (k-K)e^{-2kh}\right\} - (k-K)\left\{(k+K)e^{-2kH} - (k-K\sigma_1)\right\}]k + K/H(k),$$

and the path of integration is indented to pass beneath the poles of the above six integrands at $k = K, k = k_1$ and $k = k_2$.

The multipoles (30) can be expanded in terms of polar co-ordinates and we obtain

$$\phi^{IIIs,a}_n = \frac{\cos n\theta, \sin n\theta}{r^n} + \sum_{m=0}^{\infty} A_{nm}r^m\cos m\theta, \sin m\theta, \tag{31}$$

where

$$A_{nm} = \frac{(-1)^{n+m}}{(n-1)!m!}\int_0^\infty k^{n+m-1}E(k)e^{ky}dk. \tag{32}$$

The far-field form of the multipoles, in the lower layer, is given by

$$\phi^{IIIs,a}_n \sim (\pi i, \mp\pi)\frac{(-1)^n}{(n-1)!}(K^{n-1}E^Ke^{\pm iKx+Ky} + k_1^{n-1}E^{k_1}e^{\pm ik_1x+k_1y} + k_2^{n-1}E^{k_2}e^{\pm ik_2x+k_2y}), \tag{33}$$

as $x = \pm\infty$. Here E^K, E^{k_1}, E^{k_2} are the residues of $E(k)$ at $k = K, k = k_1$ and $k = k_2$ respectively, given by

$$E^K = 2K^3\frac{(1+\sigma_1)(1+\sigma_2)e^{Kf}e^{-2K(h+H)}}{h(K)},$$

$$E^{k_j} = e^{k_jf}[(k_j+K\sigma_2)\{(k_j+K\sigma_1)e^{-2k_j(h+H)} - (k_j-K)e^{-2k_jh} - (k_j-K)\{(k_j+K)e^{-2k_jH} - (k_j-K\sigma_1)]k_j + K/H'(k_j), \quad j = 1, 2.$$

2.1.1. Incident wave train of wavenumber K

Let us consider the case of a wave train of wavenumber K and the incident wave potential is of the form $\phi_{inc}^{III} = e^{iKx+Ky}$, when expanded about $r = 0$, has the form

$$\phi_{inc}^{III} = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!} K^m r^m e^{Kf} (\cos m\theta - i \sin m\theta). \tag{33}$$

To solve this scattering problem we write

$$\phi_K = \phi_{inc} + \sum_{n=1}^{\infty} a^n \left(a_n \phi_n^a + b_n \phi_n^s \right), \tag{34}$$

where a_n and b_n are unknown constants to be determined.

To solve for a_n and b_n the polar expansions of the multipoles (31) and the incident wave (33) are substituted into (34) and applying the body boundary condition $\frac{\partial \phi_K^{III}}{\partial r} = 0$ on $r = a$ and using the orthogonal properties of the trigonometric functions, obtain two infinite systems of linear equations for unknowns a_n and b_n which are

$$(a_m, b_m) - \sum_{n=1}^{\infty} a^{n+m} A_{nm} (a_n, b_n) = (-i, 1) \frac{(-Ka)^m}{m!} e^{Kf}, \quad m = 1, 2, ... \tag{35}$$

Since left-hand sides of the systems of equations are of the same nature and the right-hand sides of the systems differ by a factor $-i$, we find that

$$a_n = -i b_n.$$

Eq. (35) is solved by truncation 5×5 systems to produce the numerical results.

Thus ϕ_K^{III} is obtained as

$$\phi_K^{III} = \phi_{inc}^{III} + \sum_{n=1}^{\infty} a^n b_n \left(\phi_n^{IIIs} - i \phi_n^{IIIa} \right). \tag{36}$$

Using far-field form of the multipoles, in the lower layer, then we get $\phi_K^{III} \sim \phi_{inc}^{III}, \quad \text{as} \quad x \to -\infty.$

The far-field form for ϕ_K^{III} in the lower fluid, can be written as

$$\phi_K^{III} \sim \begin{cases} e^{iKx+Ky} + R_K^K e^{-iKx+Ky} + \sum_{j=1}^{2} R_{k_j}^K e^{-ik_j x + k_j y}, & \text{as } x \to -\infty, \\ T_K^K e^{iKx+Ky} + \sum_{j=1}^{2} T_{k_j}^K e^{ik_j x + k_j y}, & \text{as } x \to \infty. \end{cases}$$

Using (36) we can obtain the reflection and transmission coefficients:

$$R_K^K = R_{k_1}^K = R_{k_2}^K \equiv 0, \tag{37}$$

$$T_K^K = 1 + 2\pi i \sum_{n=1}^{\infty} \frac{(-1)^n}{(n-1)!} a^n K^{n-1} E^K b_n, \tag{38}$$

$$T_{k_j}^K = 2\pi i \sum_{n=1}^{\infty} \frac{(-1)^n}{(n-1)!} a^n k_j^{n-1} E^{k_j} b_n, \quad j = 1, 2. \tag{39}$$

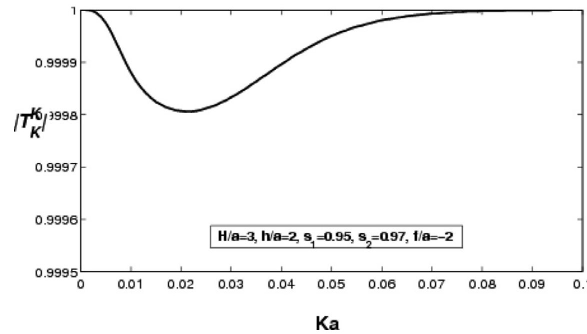

Fig. 1. Transmission coefficient due to a wave of wave number K incident on the cylinder in the lower layer.

2.1.2. Incident wave train of wavenumber k_j, $j = 1, 2$

For an incident wave of wave number k_j the mathematical analysis is the same except that K is to be replaced by k_j in the above equations. Also the far-field forms of $\phi_{k_j}^{III}$ in the lower layer, can be written as

$$\phi_{k_j}^{III} \sim \begin{cases} e^{ik_j x + k_j y} + R_K^{k_j} e^{-iKx+Ky} + \sum_{s=1}^{2} R_{k_s}^{k_j} e^{-ik_s x + k_s y}, & \text{as} \quad x \to -\infty, \\ T_K^{k_j} e^{iKx+Ky} + \sum_{s=1}^{2} T_{k_s}^{k_j} e^{ik_s x + k_s y}, & \text{as} \quad x \to \infty. \end{cases}$$

Here also we find that the reflection coefficients $R_K^{k_j}$, $R_{k_1}^{k_j}$ and $R_{k_2}^{k_j}$ are identically zero. For the transmission coefficients we obtain

$$T_K^{k_j} = 2\pi i \sum_{n=1}^{\infty} \frac{(-1)^n}{(n-1)!} a^n K^{n-1} E^K b_n, \quad j = 1, 2, \tag{40}$$

$$T_{k_s}^{k_j} = \delta_s^j + 2\pi i \sum_{n=1}^{\infty} \frac{(-1)^n}{(n-1)!} a^n k_s^{n-1} E^{k_s} b_n, \quad j = 1, 2, \quad s = 1, 2. \tag{41}$$

In a single layer fluid of infinite depth it is well known that the reflection coefficient for a submerged horizontal cylinder is zero. This was discover by Dean [23]. Also in a two-layer fluid with lower layer having infinite depth it is well known that the reflection coefficients for a submerged circular cylinder in lower layer are zero for any cases of incident wave train [3] but they also seen that it is not the case for a cylinder in the upper layer of a two layer fluid. Here when the cylinder is in the lower layer it is seen that the reflection coefficients are zero but it will seen that it is not the cases for a cylinder in either the middle or in the upper layer of a three-layer fluid.

2.1.3. Numerical results

Figs. 1 and 2 show the transmission coefficients for the case of an incident wave of wavenumber K incident on a circular cylinder in the lower layer. Similarly Figs. 3, 4 and Figs. 5, 6 show the transmission coefficients due to a wave of wave numbers k_1 and k_2, respectively incident on a cylinder in the lower layer. In all those plots the immersion depth $f/a = -2$,the depth of the middle fluid h/a is 2, depth of

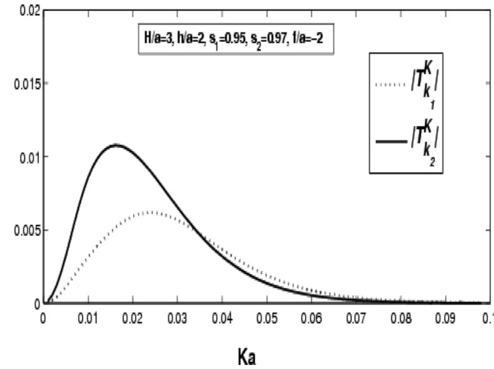

Fig. 2. Transmission coefficient due to a wave of wave number K incident on the cylinder in the lower layer.

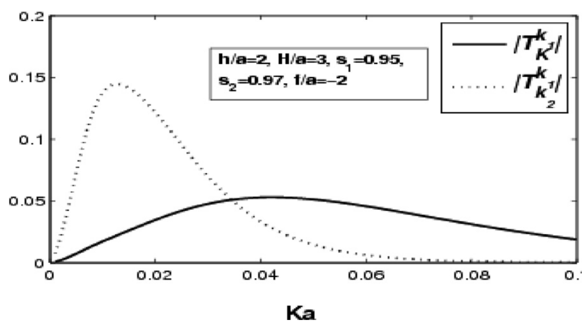

Fig. 3. Transmission coefficient due to a wave of wave number k_1 incident on the cylinder in the lower layer.

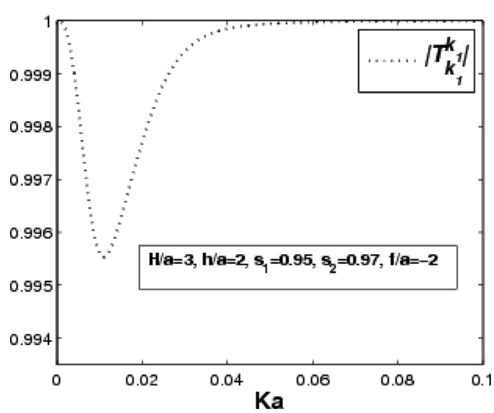

Fig. 4. Transmission coefficient due to a wave of wave number k_1 incident on the cylinder in the lower layer.

the upper fluid H/a is 3 and the density ratios s_1 and s_2 are 0.95 and 0.97, respectively.

The transmission coefficient $|T_K^K|$ corresponding to the wavenumber K shown in Fig. 1 first decreases as Ka increases for low to moderate values of Ka but it increases as Ka further increases. Fig. 2 describes the behavior of $|T_{k_1}^K|$, $|T_{k_2}^K|$, the transmission coefficients corresponding to the wavenumber k_1 and k_2 respectively which are complementary to the behavior of $|T_K^K|$ and are very small in comparison to $|T_K^K|$, but show that there is some conversion of energy from one wavenumber to the other. Similarly the transmission coefficients $|T_{k_1}^{k_1}|$, $|T_{k_2}^{k_2}|$ corresponding to the wave of wavenumber k_1 due to a wave of wavenumber k_1 incident on a cylinder and k_2 due to a wave of wavenumber k_2 incident on cylinder show in Figs. 4 and 6 respectively also first decrease as Ka increases for low to moderate values of Ka but they increase as Ka further increases. Fig. 3 describes the behavior of $|T_K^{k_1}|$, $|T_{k_2}^{k_1}|$, the transmission coefficients of wave of wavenumbers K and k_2 respectively due to a wave of wavenumber k_1 incident on a cylinder which are complementary to the behavior of $|T_{k_1}^{k_1}|$ (Fig. 4). Similarly Fig. 5 describes the behavior of $|T_K^{k_2}|$, $|T_{k_1}^{k_2}|$, the transmission coefficients of wave of wavenumbers K and k_1 respectively due to a wave of wavenumber k_2 incident on a cylinder which are complementary to the behavior of $|T_{k_2}^{k_2}|$ (Fig. 6). All the numerical values of the transmission coefficients have been checked for their correctness form the energy identities.

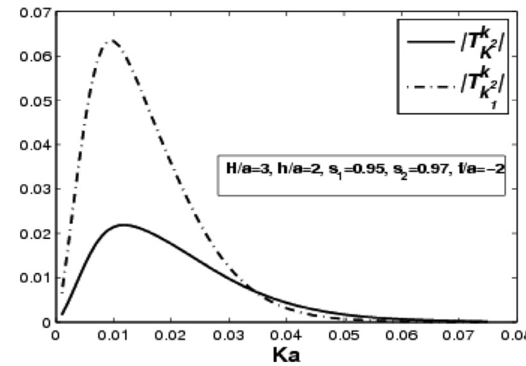

Fig. 5. Transmission coefficient due to a wave of wave number k_2 incident on the cylinder in the lower layer.

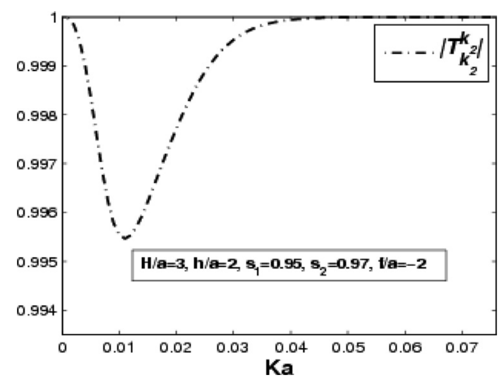

Fig. 6. Transmission coefficient due to a wave of wave number k_2 incident on the cylinder in the lower layer.

If we let $s_1 \to 0$ in the problem (corresponding to $\sigma_1 \to 1$) then we see that the multipoles defined by (29) and (30) go over to the two-layer multipoles evaluated by [3]. Thus by letting $s_1 \to 0$ in the above analysis we recover the results for the scattering by a horizontal circular cylinder in two-layer fluid with lower layer having infinite depth [3].

2.2. Cylinder in the middle layer

A horizontal circular cylinder of radius a has its axis at $y = f(> 0)$ and its generator runs parallel to the z-axis

($f/a > 1$). Polar co-ordinates are again defined via (27) and suitable multipoles, satisfying conditions (2)–(7), have the forms

$$\phi_n^{Is,a} = \frac{1}{(n-1)!}\int_0^\infty k^{n-1} \times \left(A_M^{(0,1)}(k)e^{ky} + B_M^{(0,1)}(k)e^{-ky}\right)\cos kx, \sin kx\, dk, \quad (42)$$

$$\phi_n^{IIs,a} = \frac{\cos n\theta, \sin n\theta}{r^n} + \frac{1}{(n-1)!}\int_0^\infty k^{n-1}\big(C_M^{(0,1)}(k)e^{ky} + D_M^{(0,1)}(k)e^{-ky}\big)\cos kx, \sin kx\, dk, \quad (43)$$

$$\phi_n^{IIIs,a} = \frac{1}{(n-1)!}\int_0^\infty k^{n-1}E_M^{(0,1)}(k)e^{ky}\cos kx, \sin kx\, dk, \quad (44)$$

where

$$A_M^{(m)}(k) = 2\frac{(k+K)Ke^{-2k(h+H)}}{(1-s_1)H(k)}[(-1)^{n+m}(K\sigma_2 - k)e^{kf} - (k-K)e^{-kf}], \quad m = 0, 1,$$

$$B_M^{(m)}(k) = \frac{(k-K)e^{2k(h+H)}}{k+K}A_M^{(m)}(k),$$

$$C_M^{(m)}(k) = \frac{(k-K)e^{-2kh} - (k+K\sigma_1)e^{-2k(h+H)}}{H(k)} \times [(k+K)\{(k-K)e^{-kf} - (-1)^{n+m}(K\sigma_2 - k)e^{kf}\}], \quad m = 0, 1,$$

$$D_M^{(m)}(k) = (-1)^{n+m}(k+K)e^{kf} \times \frac{(k-K)e^{-2kh} - (k+K\sigma_1)e^{-2k(h+H)}}{h(k)} - (k-K)e^{-kf}\frac{(K\sigma_1 - k) + (k+K)e^{-2kh}}{h(k)}, \quad m = 0, 1,$$

$$E_M^{(m)}(k) = e^{-kf} + C_M^{(m)}(k) - D_M^{(m)}(k),$$

and the path of integration is indented to pass beneath the poles of the above six integrands at $k = K$, $k = k_1$, and $k = k_2$.

The far-field form of these multipoles, in the lower fluid layer, is given by

$$\phi_n^{IIIs,a} \sim (\pi i, \pm\pi)\frac{(-1)^n}{(n-1)!}(K^{n-1}E_M^{(0,1)K}e^{\pm iKx+Ky} + k_1^{n-1}E_M^{(0,1)k_1}e^{\pm ik_1x+k_1y} + k_2^{n-1}E_M^{(0,1)k_2}e^{\pm ik_2x+k_2y}), \quad (45)$$

as $x = \pm\infty$, where $E_M^{(0,1)K}$, $E_M^{(0,1)k_1}$, $E_M^{(0,1)k_2}$ are the residues of $E_M^{(m)}(k)$ at $k = K, k = k_1$ and $k = k_2$ respectively, given by

$$E_M^{(m)K} = 2(-1)^{n+m+1}K^3(1+\sigma_1)(1-\sigma_2)e^{Kf-2K(h+K)}/H'(K), \quad m = 0, 1,$$

$$E_M^{(m)k_j} = \frac{(k_j-K)e^{-2k_jh} - (k_j+K\sigma_1)e^{-2k_j(h+H)}}{H'(k_j)} \times [(k_j+K)\{(k_j-K)e^{-k_jf} - (-1)^{n+m}(K\sigma_2 - k_j)e^{k_jf}\}] - (-1)^{n+m}(k_j-K)(k_j+K)e^{k_jf} \times \frac{(k_j-K)e^{-2k_jh} - (k_j+K\sigma_1)e^{-2k_j(h+H)}}{H'(k_j)} + (k_j-K)^2e^{-k_jf}\frac{(K\sigma_1 - k_j) + (k_j+K)e^{-2k_jh}}{H'(k_j)},$$
$$m = 0, 1, \quad j = 1, 2.$$

The polar expansions of the multipoles, similar to the case when cylinder is in the lower fluid, are

$$\phi_n^{IIs,a} = \frac{\cos n\theta, \sin n\theta}{r^n} + \sum_{m=0}^{\infty} B_{nm}^{s,a}r^m\cos m\theta, \sin m\theta, \quad (46)$$

where

$$B_{nm}^{s,a} = \frac{1}{(n-1)!m!}\int_0^\infty k^{n+m-1}((-1)^{m,m+1}C_M^{(0,1)}(k)e^{kf} + D_M^{(0,1)}(k)e^{-kf})dk. \quad (47)$$

Note that unlike the case of multipoles singular in the lower layer, the coefficients in the polar expansions of ϕ_n^s and ϕ_n^a are not the same.

2.2.1. Incident wave train of wavenumber K

For this problem ϕ_{inc}^{II} is given, in the middle fluid, by e^{iKx+Ky}.The polar expansion is given in (33) and the velocity potential ϕ_K^{II} is expanded similar as (34), where ϕ_n^s and ϕ_n^a are the symmetric and antisymmetric multipoles developed for the middle fluid respectively. After applying the body boundary condition, $\frac{\partial\phi_K^{II}}{\partial r} = 0$ on $r = a$ and also using the orthogonal properties of trigonometric functions, we obtain the two infinite system of linear equations

$$(a_m, b_m) - \sum_{n=1}^{\infty} a^{n+m}B_{nm}^{a,s}(a_n, b_n) = (-i, 1)\frac{(-ka)^m}{m!}e^{Kf}, \quad m = 1, 2\ldots \quad (48)$$

These equations were solved by truncations to 5×5 systems to produce the numerical results. The reflection and transmission coefficients can be extracted from the far-field form of the potential ϕ_K^{III},using (33) and (45) with far field form of ϕ_K^{III}, and are given by

$$R_K^K, T_K^K = (0, 1) + \pi\sum_{n=1}^{\infty}\frac{1}{(n-1)!}a^nK^{n-1}\{ib_nE_M^{(0)K} \mp a_nE_M^{(1)K}\}, \quad (49)$$

$$R_K^{k_j}, T_K^{k_j} = \pi\sum_{n=1}^{\infty}\frac{1}{(n-1)!}a^nk_j^{n-1}\{ib_nE_M^{(0)k_j} \mp a_nE_M^{(1)k_j}\}, \quad j = 1, 2. \quad (50)$$

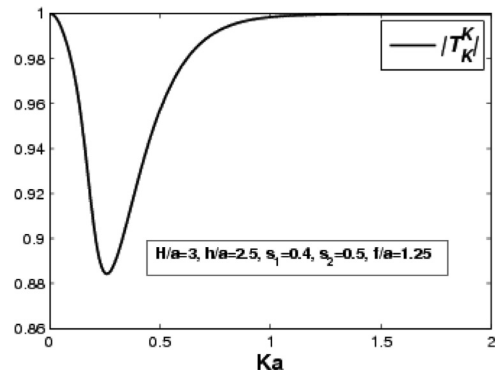

Fig. 7. Transmission coefficient due to a wave of wave number K incident on the cylinder in the middle layer.

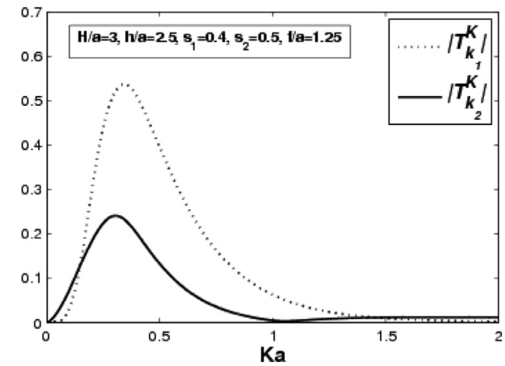

Fig. 8. Transmission coefficient due to a wave of wave number K incident on the cylinder in the middle layer.

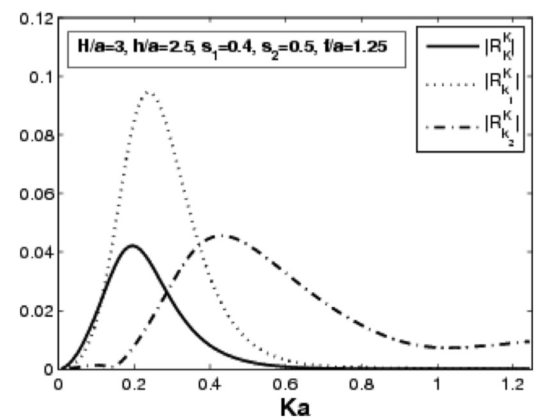

Fig. 9. Reflection coefficient due to a wave of wave number K incident on the cylinder in the middle layer.

2.2.2. Incident wave train of wavenumber k_j, $j = 1, 2$

For this problem ϕ_{inc}^{II} is given, in the middle fluid, by $e^{ik_jx}g_2^j(y)$. The polar expansion of ϕ_{inc}^{II} is given by

$$\phi_{inc}^{II} = \sum_{m=0}^{\infty}(M_1(k_j)\cos m\theta + iM_2(k_j)\sin m\theta)\frac{(k_jr)^m}{m!},$$
$$j = 1, 2, \tag{51}$$

where

$$M_{1,2}(k_j) = (-1)^{m,m+1}\frac{k_j - K\sigma_2}{K(1-s_1)(1-\sigma_2)}e^{k_jf} + \frac{k_j - K}{K(1-s_1)(1-\sigma_2)}e^{-k_jf}.$$

The velocity potential $\phi_{k_j}^{II}$ for this scattering problem can again be expanded in multipoles similar to (33) and the equations for a_n and b_n are

$$(a_m, b_m) - \sum_{n=1}^{\infty}a^{n+m}B_{nm}^{a,s}(a_n, b_n) = (-i, 1)\frac{(k_ja)^m}{m!}M_{2,1}(k_j),$$
$$m = 1, 2, \ldots \tag{52}$$

The reflection and transmission coefficients can be extracted from the far-field form of the potential $\phi_{k_j}^{III}$ using (33) and (45) with far field form of ϕ_K^{III}. The expressions for $R_{k_j}^K$ and $R_{k_j}^{k_{1,2}}$ are similar to (49) and (50) with appropriate changes, and the transmission coefficients are given by

$$T_{k_j}^K, T_{k_j}^{k_s} = \pi\sum_{n=0}^{\infty}\frac{1}{(n-1)!}a^n\left(K^{n-1}, k_j^{n-1}\right)\left\{ib_nE_M^{(0)K,k_j} + a_nE_M^{(1)K,k_j}\right\} + (0, \delta_s^j),\ j = 1, 2,\ s = 1, 2. \tag{53}$$

2.2.3. Numerical results

We choose $h/a = 2.5$, $H/a = 3$, $f/a = 1.25$, $s_1 = 0.4$, $s_2 = 0.5$ for which the transmission and reflection coefficients due to an incident wave of wave number K are depicted in Figs. 7–9. Figs. 10–12 and Figs. 13–15 show the results for the same parameters but an incident wave of wave number k_1 and k_2, respectively. Fig. 7 shows that the transmission coefficients $|T_K^K|$ first decreases as Ka increases, attains a minimum value and then increases as Ka further increases. Figs. 8 and 9 show that the transmission coefficients $|T_{k_1}^K|$, $|T_{k_2}^K|$ and the reflection coefficients $|R_K^K|$, $|R_{k_1}^K|$, $|R_{k_2}^K|$ respectively, first increases as Ka increases, each attains a maximum values and the decreases as Ka further increases. The transmission coefficients of waves of wavenumbers k_1 and k_2 and the reflection coefficients of wave of wavenumbers K, k_1, k_2 for an incident wave of wavenumber K, shown in Figs. 8 and 9 respectively, are smaller in comparison to those for wave of wavenumber K, but their non-zero values show that there is some conversion of energy from one wave number to the other.

Figs. 10 and 13 show that the transmission coefficients $|T_{k_1}^{k_1}|$, $|T_{k_2}^{k_2}|$ of wave of wavenumbers k_1 and k_2 for incident wave of wavenumbers k_1 and k_2, respectively, Figs. 11 and 12 show that the transmission coefficients $|T_K^{k_1}|$, $|T_{k_2}^{k_1}|$ of wave of wavenumbers K and k_2 and the reflection coefficients $|R_K^{k_1}|$, $|R_{k_1}^{k_1}|$, $|R_{k_2}^{k_1}|$ of wave of wavenumbers K, k_1, k_2 respectively for an incident wave of wavenumber k_1. Figs. 14 and 15 show that the transmission coefficients $|T_K^{k_2}|$, $|T_{k_1}^{k_2}|$ of wave of wavenumbers K and k_1 and the reflection coefficients $|R_K^{k_2}|$, $|R_{k_1}^{k_2}|$, $|R_{k_2}^{k_2}|$ of wave of wavenumbers K, k_1, k_2 respectively for an incident wave of wavenumber k_2. They are somewhat similar to those

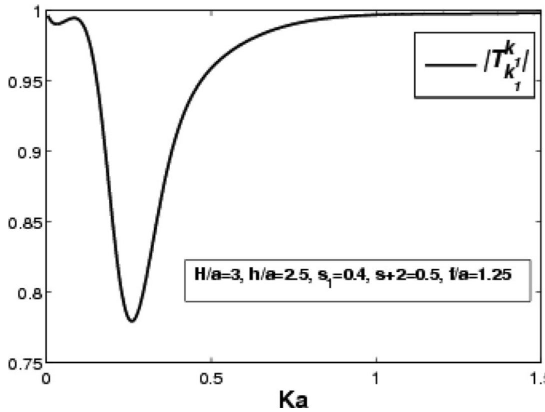

Fig. 10. Transmission coefficient due to a wave of wave number k_1 incident on the cylinder in the middle layer.

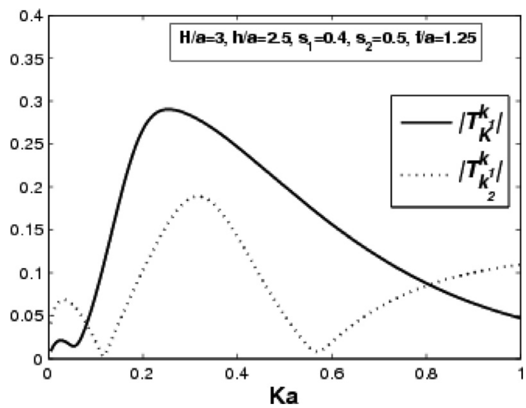

Fig. 11. Transmission coefficient due to a wave of wave number k_1 incident on the cylinder in the middle layer.

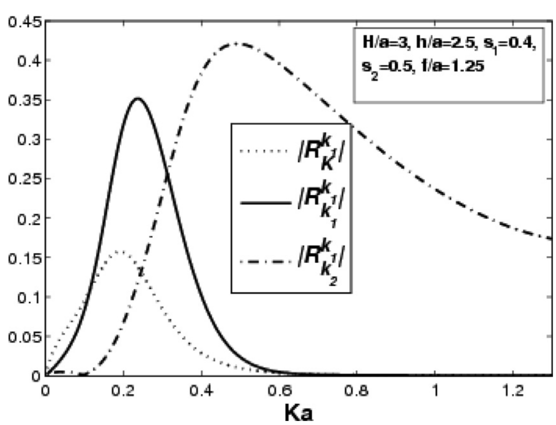

Fig. 12. Reflection coefficient due to a wave of wave number k_1 incident on the cylinder in the middle layer.

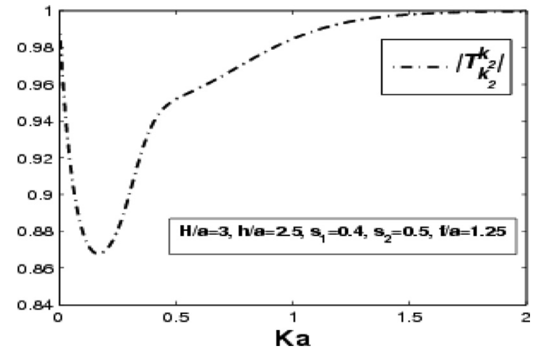

Fig. 13. Transmission coefficient due to a wave of wave number k_2 incident on the cylinder in the middle layer.

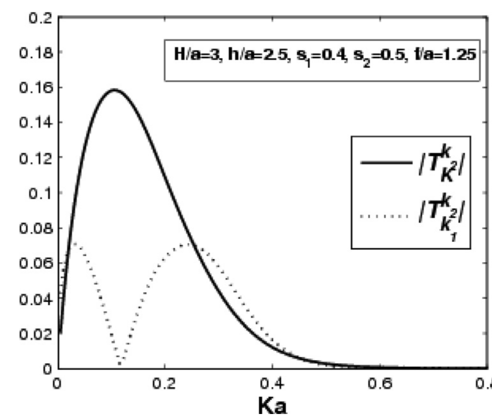

Fig. 14. Transmission coefficient due to a wave of wave number k_2 incident on the cylinder in the middle layer.

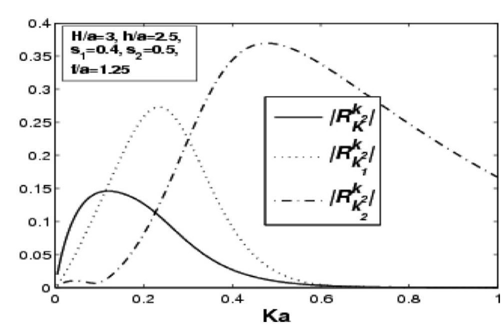

Fig. 15. Reflection coefficient due to a wave of wave number k_2 incident on the cylinder in the middle layer.

for the scattering of an incident wave of wavenumber K by a circular cylinder in the middle layer and display the same characteristics.

2.3. Cylinder in the upper layer

A horizontal circular cylinder of radius a has its axis at $y = f + h(> 0)$and its generator runs parallel to the z-axis ($f/a > 1$). Polar co-ordinates are again defined via (27) and suitable multipoles, satisfying conditions (2)–(7), have the forms

$$\phi_n^{Is,a} = \frac{\cos n\theta, \sin n\theta}{r^n} + \frac{1}{(n-1)!}\int_0^\infty k^{n-1} \times\left(A_U^{(0,1)}(k)e^{ky} + B_U^{(0,1)}(k)e^{-ky}\right)\cos kx, \sin kx\, dk, \quad (54)$$

$$\phi_n^{IIs,a} = \frac{1}{(n-1)!}\int_0^\infty k^{n-1}(C_U^{(0,1)}(k)e^{ky} + D_U^{(0,1)}(k)e^{-ky})\cos kx, \sin kx\, dk, \quad (55)$$

$$\phi_n^{IIIs,a} = \frac{1}{(n-1)!}\int_0^\infty k^{n-1}E_U^{(0,1)}(k)e^{ky}\cos kx, \sin kx\, dk, \quad (56)$$

where

$$A_U^{(m)}(k) = (k-K)e^{-kf-2kH}[-(k+K\sigma_1)e^{-2kh} + (k+K\sigma_2)] \\ +(-1)^{n+m}e^{-2k(h+H)+kf}[(k-K\sigma_1)(k-K\sigma_2) \\ -(k+K)(k-K)e^{-2kh}]\frac{(k+K)}{H(k)}, m=0,1,$$

$$B_U^{(m)}(k) = [(-1)^{n+m}(k+K)e^{kf-2kH} + (k-K)e^{-kf+2kH}] \\ \times \frac{(k-K\sigma_2)-(k+K\sigma_1)e^{-2kh}}{h(k)}, \quad m=0,1,$$

$$C_U^{(m)}(k) = -2Ks_1(k-K\sigma_2) \\ \times \frac{(-1)^{n+m}(k+K)e^{-2k(h+H)+kf} + (k-K)e^{-kf}}{(1-s_1)H(k)}, \\ m=0,1,$$

$$D_U^{(m)}(k) = \frac{(k-K)C_U^{(m)}(k)}{(k-K\sigma_2)}, E_U^{(m)}(k) = C_U^{(m)}(k) - D_U^{(m)}(k), \\ m=0,1,$$

and the path of integration is indented to pass beneath the poles of the above six integrands at K, k_1, k_2.

The far-field form of these multipoles, in the lower fluid layer, is given by

$$\phi_n^{IIIs,a} \sim (\pi i, \pm\pi)\frac{1}{(n-1)!}(K^{n-1}E_U^{(0,1)K}e^{\pm iKx+Ky} \\ + k_1^{n-1}E_U^{(0,1)k_1}e^{\pm ik_1x+k_1y} + k_2^{n-1}E_U^{(0,1)k_2}e^{\pm ik_2x+k_2y}), \quad (57)$$

as $x = \pm\infty$, where$E_U^{(0,1)K}$, $E_U^{(0,1)k_1}$, $E_U^{(0,1)k_2}$are the residues of $E_U^{(m)}(k)$ at $k=K, k=k_1$ and $k=k_2$ respectively, given by

$$E_U^{(m)K} = -\frac{4(-1)^{n+m}K^3s_1(1-\sigma_2)e^{Kf-2K(h+K)}}{(1-s_1)h(K)}, \quad m=0,1,$$

$$E_U^{(m)k_j} = \frac{(-1)^{n+m}(k_j+K)e^{-2k_j(h+H)+k_jf} + (k_j-K)e^{-k_jf}}{(1-s_1)H'(k_j)} \\ \times 2Ks_1\{-(k_j-K\sigma_2)+(k_j-K)\}, m=0,1, \\ j=1,2.$$

The polar expansions of the multipoles, similar to the case when cylinder is in the middle fluid, are

$$\phi_n^{Is,a} = \frac{\cos n\theta, \sin n\theta}{r^n} + \sum_{m=0}^{\infty} C_{nm}^{s,a} r^m \cos m\theta, \sin m\theta, \quad (58)$$

where

$$C_{nm}^{s,a} = \frac{1}{(n-1)!m!}\int_0^{\infty} k^{n+m-1}((-1)^{m,m+1}A_U^{(0,1)}(k)e^{kf} \\ + B_U^{(0,1)}(k)e^{-kf})dk. \quad (59)$$

Note that unlike the case of multipoles singular in the lower layer, the coefficients in the polar expansions of ϕ_n^s and ϕ_n^a are not the same.

2.3.1. Incident wave train of wavenumber K

For this problem ϕ_{inc}^I is given, in the upper fluid, by e^{iKx+Ky}.The polar expansion is given in (33) and the velocity potential ϕ_K^I is expanded similar to as (34), where ϕ_n^s and ϕ_n^a are the symmetric and antisymmetric multipoles developed for the upper fluid respectively. After applying the body boundary condition, $\frac{\partial \phi_K^I}{\partial r} = 0$ on $r=a$ and also using the orthogonal properties of trigonometric functions, we obtain the two infinite systems of linear equations

$$(a_m, b_m) - \sum_{n=1}^{\infty} a^{n+m} C_{nm}^{a,s}(a_n, b_n) = (-i, 1)\frac{(-Ka)^m}{m!}e^{Kf}, \\ m=1,2,... \quad (60)$$

These equations were solved by truncations to 5×5 systems to produce the numerical results. The reflection and transmission coefficients can be extracted from the far-field form of the potential ϕ_K^{III}, using (33) and (57) with far field form of ϕ_K^{III}, and are given by

$$R_K^K, T_K^K = (0,1) + \pi\sum_{n=1}^{\infty}\frac{1}{(n-1)!}a^nK^{n-1}\{ib_nE_U^{(0)K} \mp a_nE_U^{(1)K}\}, \quad (61)$$

$$R_K^{k_j}, T_K^{k_j} = \pi\sum_{n=1}^{\infty}\frac{1}{(n-1)!}a^nk_j^{n-1}\{ib_nE_U^{(0)k_j} \mp a_nE_U^{(1)k_j}\}, \\ j=1,2. \quad (62)$$

2.3.2. Incident wave train of wavenumber $k_j, j=1,2$

For this problem ϕ_{inc}^I is given, in the upper fluid, by $e^{ik_jx}g_1^j(y)$. The polar expansion of ϕ_{inc}^I is given by

$$\phi_{inc}^I = \sum_{m=0}^{\infty}(N_1(k_j)\cos m\theta + iN_2(k_j)\sin m\theta)\frac{(k_jr)^m}{m!}, \quad j=1,2, \quad (63)$$

where

$$N_{1,2}(k_j) = 2(-1)^{m,m+1}\frac{(k_j+K)e^{-2k_j(h+K)}}{N(k_j)}e^{k_jf} \\ + \frac{k_j-K}{N(k_j)}e^{-k_jf}, \quad j=1,2,$$

$$N(k_j) = (1-s_1)(1-\sigma_2)[(k_j+K)e^{-2k_jH} - (k_j-K\sigma_1)], \\ j=1,2.$$

The velocity potential $\phi_{k_j}^I$ for this scattering problem can again be expanded in multipoles similar to (33) and the equations for a_n and b_n are

$$(a_m, b_m) - \sum_{n=1}^{\infty} a^{n+m}C_{nm}^{a,s}(a_n, b_n) = (i,1)\frac{(k_ja)^m}{m!}N_{2,1}(k_j), \\ m=1,2,... \quad (64)$$

The reflection and transmission coefficients can be extracted from the far-field form of the potential $\phi_{k_j}^{III}$ using (33)

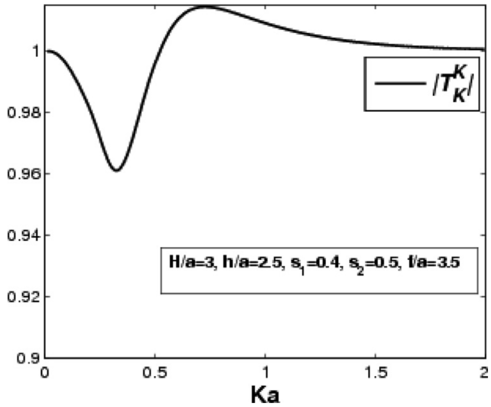

Fig. 16. Transmission coefficient due to a wave of wave number K incident on the cylinder in the upper layer.

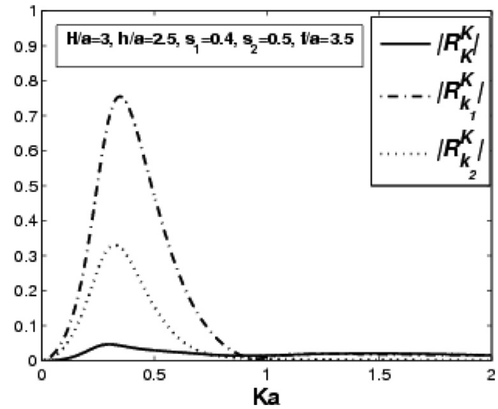

Fig. 18. Reflection coefficient due to a wave of wave number K incident on the cylinder in the upper layer.

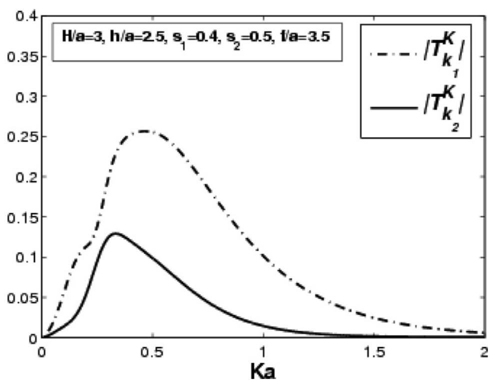

Fig. 17. Transmission coefficient due to a wave of wave number K incident on the cylinder in the upper layer.

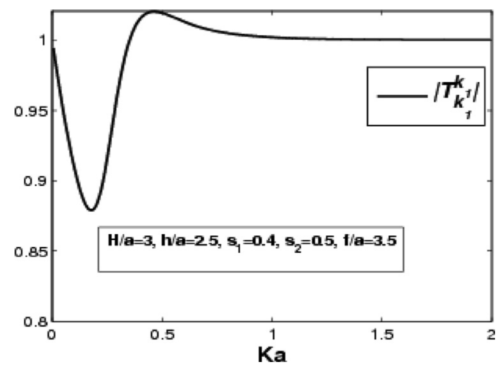

Fig. 19. Transmission coefficient due to a wave of wave number k_1 incident on the cylinder in the upper layer.

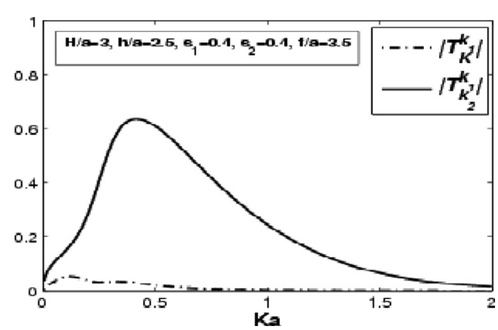

Fig. 20. Transmission coefficient due to a wave of wave number k_1 incident on the cylinder in the upper layer.

and (57) with far field form of ϕ_K^{III}. The expressions for $R_{k_j}^K$ and $R_{k_j}^{k_{1,2}}$ are similar to (61) and (62) with appropriate changes, and the transmission coefficients are given by

$$\begin{aligned} T_{k_j}^K, T_{k_j}^{k_s} = \pi \sum_{n=0}^{\infty} \frac{1}{(n-1)!} a^n (K^{n-1}, k_j^{n-1}) \\ \times \left\{ i b_n E_U^{(0)K,k_j} + a_n E_U^{(1)K,k_j} \right\} \\ + (0, \delta_s^j), \quad j = 1, 2, \mathrm{s} = 1, 2. \end{aligned} \tag{65}$$

2.3.3. Numerical results

We choose $h/a = 2.5$, $H/a = 3$, $f/a = 3.5$, $s_1 = 0.4$, $s_2 = 0.5$ for which the transmission and reflection coefficients due to an incident wave of wave number K are depicted in Figs. 16–18. Figs. 19–21 and Figs. 22–24 show the results for the same parameters but an incident wave of wave number k_1 and k_2 respectively. Fig. 16 shows that the transmission coefficients $|T_K^K|$ first decreases as Ka increases, attains a minimum value and then increases as Ka further increases. Figs. 17 and 18 show that the transmission coefficients $|T_{k_1}^K|$, $|T_{k_2}^K|$and the reflection coefficients $|R_K^K|$, $|R_{k_1}^K|$, $|R_{k_2}^K|$ respectively, first increases as Ka increases, each attains a maximum values and the decreases as Ka further increases. The transmission coefficients of waves of wavenumbers k_1 and k_2 and the reflection coefficients of wave of wavenumbers K, k_1, k_2 for an incident wave of wavenumber K, shown in Figs. 17 and 18 respectively, are smaller in comparison to those for wave of wavenumber K, but their non-zero values show that there is some conversion of energy from one wave number to the other.

Figs. 19 and 20 show that the transmission coefficients $|T_{k_1}^{k_1}|$, $|T_{k_2}^{k_2}|$ of wave of wavenumbers k_1 and k_2 for incident wave of wavenumbers k_1 and $k_{2,}$ respectively, Figs. 20 and 21 show that the transmission coefficients $|T_K^{k_1}|$, $|T_{k_2}^{k_1}|$of wave of wavenumbers K and k_2 and the reflection coefficients $|R_K^{k_1}|$, $|R_{k_1}^{k_1}|$, $|R_{k_2}^{k_1}|$ of wave of wavenumbers K, k_1, k_2 respectively for an incident wave of wavenumber k_1. Figs. 23 and 24 show that

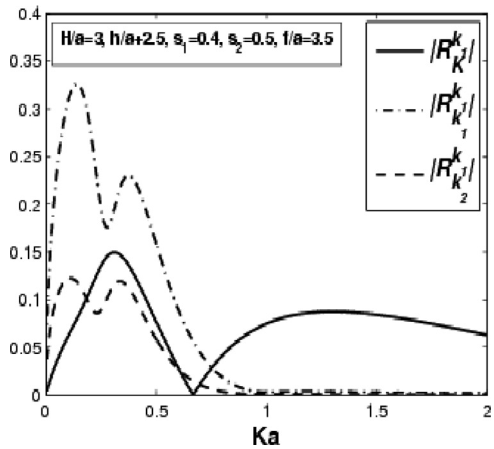

Fig. 21. Reflection coefficient due to a wave of wave number k_1 incident on the cylinder in the upper layer.

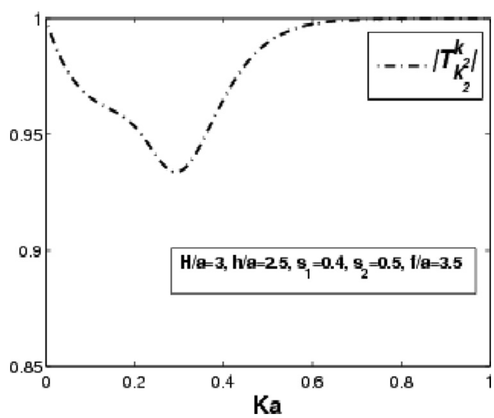

Fig. 22. Transmission coefficient due to a wave of wave number k_2 incident on the cylinder in the upper layer.

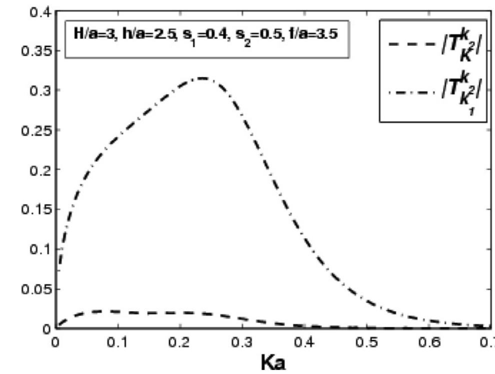

Fig. 23. Transmission coefficient due to a wave of wave number k_2 incident on the cylinder in the upper layer.

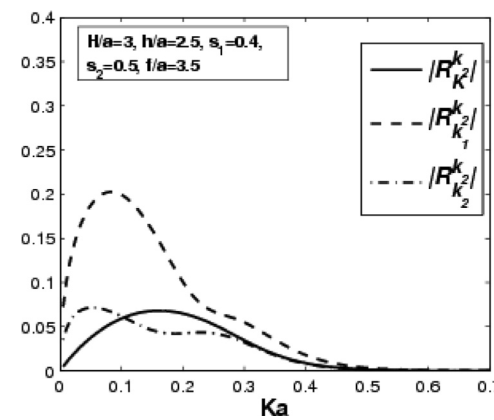

Fig. 24. Reflection coefficient due to a wave of wave number k_2 incident on the cylinder in the upper layer.

the transmission coefficients $|T_K^{k_2}|$, $|T_{k_1}^{k_2}|$ of wave of wavenumbers K and k_1 and the reflection coefficients $|R_K^{k_2}|$, $|R_{k_1}^{k_2}|$, $|R_{k_2}^{k_2}|$ of wave of wavenumbers K, k_1, k_2 respectively for an incident wave of wavenumber k_2. All the figures are somewhat similar to those for the scattering of an incident wave of wavenumber K (for Figs. 16–18), k_1 (for Figs. 19–21) and k_2 (for Figs. 22–24) by a circular cylinder in the middle layer and display the same characteristics but they are slight different from the figures given in the case of middle or lower layer.

Reflection and transmission coefficients are oscillatory in nature and all reflection coefficients for all the incident wave numbers, transmission coefficients in Figs. 17, 20, and 23 tend ultimately to zero for large Ka and also transmission coefficients in Figs. 16, 19, and 22 tend to unity for large Ka. All the numerical values of the reflection and transmission coefficients have been checked for their correctness from the energy identities.

3. Conclusions

In this paper we have studied the problem of water wave scattering by a horizontal circular cylinder submerged in either layer of a three-layer fluid. The middle layer is of finite depth and is bounded above by an upper layer of finite depth with free surface and the lower layer extends infinitely downwards. In such a situation propagating waves can exist at three different wave numbers for any frequency, first one propagates on the free surface, second one on the interface between upper and middle layer and third one on the interface between middle and lower layer. The systematic derivation, using Green's theorem, of all the energy identities has been obtained to the three-fluid case. When the cylinder is positioned in the lower layer of a two-layer fluid, it is well known that zero reflections occur for any radius of the cross-section of the cylinder and wave number and we have shown that for a cylinder in the lower layer (infinite) of a three layer fluid this is again the case, zero reflections occur for either of the possible wavenumbers. When the cylinder is in the upper or middle layer, zero reflection is not observed. The transmission and reflection coefficients are depicted graphically against the wave number in a number of figures. Energy identities are used as partial numerical checks for all the data points. A brief summary of your research results should be included in this section toward the end of the paper.

Acknowledgment

The authors thank the Editor and the Reviewers for their comments and suggestions to revise the paper.

References

[1] G.G. Stokes, Trans. Camb. Philos. Soc. 8 (1847) 441–445 Reprinted in Maths. Phys. papers, Cambridge University Press, 1847.
[2] H. Lamb, Hydrodynamics, Cambridge University Press, 1932.
[3] C.M. Linton, M. McIver, J. Fluid Mech. 304 (1995) 213–229.
[4] J.R. Cadby, C.M. Linton, J. Fluid Mech. 423 (2000) 155–173.
[5] C.M. Linton, J.R. Cadby, J. Fluid Mech. 461 (2002) 343–364.
[6] S.R Manam, T. Sahoo, J. Eng. Math. 52 (2005) 355–377.
[7] P.G. Chamberlian, D. Porter, J. Fluid Mech. 524 (2005) 207–228.
[8] I. Ten, M. Kashiwagi, J. Mar. Sci. Technol. 9 (2004) 127–141.
[9] M. Kashiwagi, I. Ten, M. Yesunaga, J. Mar. Sci. Technol. 11 (2006) 150–164.
[10] D. Das, B.N. Mandal, in: Proceedings of the 21st International Workshop on Water Waves and Floating Bodies, IWWWFB, Loughborough University, UK, 2006, pp. 29–32.
[11] D. Das, B.N. Mandal, Int. J. Eng. Sci. 45 (2007) 842–872.
[12] D. Das, Bull. Calcutta Math. Soc. 100 (2) (2008) 165–176.
[13] D. Das, B.N. Mandal, Appl. Ocean Res. 32 (2010) 358–366.
[14] D. Das, N. Thakur, Int. J. Appl. Math. Eng. Sci. 8 (2014) 45–63.
[15] H. Michallet, F. Dias, Proceedings of the 9th International Offshore and Polar Engineering Conference, (1999) pp. 193–198.
[16] G.I. Taylor, Proc. R. Soc. Lond. Ser. A 132 (1931) 499–523.
[17] A. Chakrabarti, P. Daripa, Hamsapriye, Z. Angew. Math. Phys. 56 (2005) 1084–1097.
[18] M.J. Chen, L.K. Forbes, J. Fluid Mech. 594 (2008) 157–181.
[19] R. Mondal, T. Sahoo, Z. Angew. Math. Phys. 65 (2014) 349–375.
[20] J.N. Newman, in: Proceedings of the 11th Symposium on Naval Hydrodynamics, London, 1976.
[21] D. Das, B.N. Mandal, A. Chakrabarti, Fluid Dyn. Res. 40 (2008) 253–272.
[22] S. Gradshteynt, M. Ryzhiki, Tables of Integrals, Series and Products, Academic Press, 1965.
[23] W.R. Dean, Proc. Camb. Philos. Soc. 44 (1948) 957–983.